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ABSTRACT

This volume was organized at the request of the SMSG Panel on Research for their use in identifying needed research in mathematics education. The five papers provide a review of research in each of their areas up through early 1969. Papers included are: (1) Attitudes Toward Mathematics, by Lewis R. Aiken, Jr.; (2) Classroom Teaching of Mathematics, by James T. Fey; (3) Piagetian Studies and Mathematics Learning, by D. B. Harrison; (4) Computers in Mathematics Instruction, by Larry L. Hatfield; and (5) Problem-Solving and Creative Behavior in Mathematics, by Jeremy Kilpatrick. Extensive references follow each of the papers. (RH)

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# Studies in Mathematics

## VOLUME XIX

## Reviews of Recent Research in Mathematics Education

Edited by
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L. Ray Carry, University of Texas

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# PREFACE

This volume was organized at the request of the SMSG Panel on Research for their use in identifying needed research in mathematics education. At a meeting of the Panel in May 1969, several very recent comprehensive reviews, as yet unpublished, were discussed, and the Panel decided to request copies for their use. Originally, these manuscripts were to be reproduced in enough copies for the Panel -- either separately or together in an in-house paper. In a work session at Stanford during the summer, 1969, we examined the five manuscripts collected for the Panel and were impressed. We felt these comprehensive reviews were of general interest and therefore argued for making these reviews available to a wider audience. As a result, this volume in the STUDIES IN MATHEMATICS series was prepared.

Each previous volume in the STUDIES IN MATHEMATICS series has dealt with some mathematics topic or mathematics with discussion of pedagogy and curriculum. This is the first volume to be devoted specifically to mathematics education or to research. The intent of the series, however, is to provide a set of references for persons concerned with secondary and elementary school mathematics teaching. We feel that this volume of reviews is within the intent of the series. For surely these reviews are of interest to teachers, supervisors, graduate students, and college teachers.

The five papers vary in style and emphasis. This is partly because the papers were originally prepared for diverse audiences, partly because of the nature of the respective research areas, and partly because of the different orientations of the authors. We feel a very important characteristic of these reviews is that they are up-to-date. They are reviews of material up through early 1969 in each of the areas. Taken together, these papers -- though written for different purposes and different audiences -- should give the reader a sense of some significant current activities and trends in research in mathematics education.

Aiken began his paper in connection with a seminar that met in SMSG headquarters during the academic year 1968-69 while he held an U.S. Office of Education Post-Doctoral Fellowship. It was completed during the latter part of his fellowship at the University of Georgia. The paper is a review of the research on attitudes toward mathematics during the past ten years, but includes a very useful discussion of methods of measuring attitudes toward mathematics, as an introduction, and incorporates brief statements on methodology (e.g., use of residual gain scores, use of cross-lag correlation) within the text. A slightly modified version of the paper has been submitted to the Review of Educational Research.

In his review of classroom teaching of mathematics, Fey also includes a sampling of the research in teacher effectiveness, programmed instruction, and teacher education. His review seems to draw on a large number of doctoral studies -- "methods" studies are popular thesis topics. Fey notes a rather disorganized and scattered field of research with virtually no programmatic research where studies lead from one to the next in some systematic way. There is pessimism, urgency and optimism mixed in his message: what has been done is meager and could be improved, this is an important line of disciplined inquiry with the potential for an improved mathematics instruction, and there are some guidelines provided which should lead to better studies. This review concentrates on research published during the past five years.

Harrison includes a discussion of some major theoretical points in Piagetian studies as well as a review of recent studies. He concludes his paper with a brief discussion of sample mathematics teaching approaches consonant with a Piagetian point of view. Such illustrations of a line of thought proceeding from theory through experimentation to application are all too rare in mathematics education.

Hatfield draws on the literature about computer assisted instruction more than research reports. This is appropriate to the CAI literature since it is a relatively new research area. Hatfield wrote his review for an audience that was primarily teachers of mathematics. He makes a cogent explanation of the various uses of computers in mathematics instruction and illustrates the potential for each use along with the need for programs of research.

Kilpatrick, in examining the research on problem solving and creative behaviors in mathematics, also found an absence of any programs of research by mathematics educators. There were lines of investigation suggested from psychological research, and Kilpatrick notes a need for more clinical studies to generate hypotheses for future inquiry. This review also concentrates on studies published during the past five years.

Earlier versions of the papers by Harrison, Hatfield, and Kilpatrick were presented in research reporting sessions at the 47th Annual Meeting of the National Council of Teachers of Mathematics, April 1969, in Minneapolis. The Fey and Kilpatrick papers were prepared as review chapters for the October 1969 issue of the Review of Educational Research. Owing to space limitations in the RER, the chapters had to be drastically cut. We thank Gene Glass, editor of RER for granting permission to include the full versions in this volume.

As editors, we must admit to contributing very little to this volume other than interest, enthusiasm, and a careful reading of the manuscripts. We have done only minor editing to bring some similarity of reference style to the papers. These papers are therefore more tentative in nature than would have been the case if they had passed through a system of rigorous editorial review. Because the papers are tentative, the authors would appreciate correspondence with investigators who find the material useful and who wish to help with the further refinement of the ideas presented herein. We commend the authors for being willing to share their work with a wider audience of mathematics educators.

J.W.W.
L.R.C.

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# ATTITUDES TOWARD MATHEMATICS[1]

Lewis R. Aiken, Jr.
Guilford College

In her "Review of Research on Psychological Problems in Mathematics Education" written approximately ten years ago, Rosalind Feierabend (1960) devoted about that same number of pages to research on attitudes toward mathematics. During the past decade a number of published reports of conference proceedings have been concerned with mathematics learning (e.g., Hooten, 1967; Morrisett and Vinsonhaler, 1965), but these reports do not treat in detail research on attitudes. Because the number of dissertations and published articles dealing with attitudes toward mathematics has increased geometrically since Feierabend's (1960) report, it is time to reappraise our knowledge of the topic.

It has been stated that before progressive education came on the scene more school failures were caused by arithmetic than by any other subject (Wilson, 1961). Even if progressive education has reduced the number of failures in arithmetic and mathematics, it is debatable whether modern curricula have fostered more positive attitudes toward the subjects. But how general are these negative attitudes, what causes them, and what can be done to make them more positive?

Some years ago the members of a committee formed to study problems in mathematics education asked these same questions (Dyer et al., 1956). Their main conclusion was that more information was needed in order to give adequate answers--information about biological inheritance and home background of the pupil, attitudes and training of teachers, and the content, organization, goals, and adaptability of the curriculum. A fair question is: What information on the influences of these three types of factors has research provided since 1956? The purpose of this review is to answer that question as it pertains to research over the past ten years. Feierabend's 1960 review should be consulted for a summary of earlier investigations.

The interpretation of results depends to some degree on the measuring instruments employed in the research. Therefore, the review will deal first with paper-and-pencil, observational, and other methods described in the recent literature for measuring attitudes toward mathematics. Next, studies pertaining to the distribution and stability of attitudes and the effects of attitudes on achievement in mathematics will be considered. Then, with regard to the challenge of Dyer, Kalin and Lord (1956) referred to above, findings concerning the influences on student attitudes toward mathematics of the home environment, the personality characteristics of the student, the teacher, and the school curriculum will be summarized. Next, research and discussions of techniques for developing positive attitudes and modifying negative attitudes will be reviewed. In the final section of the paper, the investigations which have been reviewed will be evaluated, and some suggestions for further research will be made.

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[1] A slightly modified version of this paper, entitled "Attitudes Toward Mathematics: A Decade of Research Reviewed," has been submitted to the Review of Educational Research.

## Methods of Measuring Attitudes Toward Mathematics

It has been maintained that there are no valid measures of attitudes toward mathematics (Morrisett and Vinsonhaler, 1965, p. 133), but the fact remains that a number of techniques--some of them quite ingenious--are available to measure such attitudes. Several of these techniques are described by Corcoran and Gibb (1961), including (a) self-report methods such as questionnaires, attitude scales, incomplete sentences, projective pictures, and essays, (b) observational methods, and (c) interviews. It is observed that although the majority of investigations have dealt with attitudes toward mathematics in general, one can also measure attitudes toward specific courses or types of mathematics problems.

### Observation and Interview

Observation is superficially the most objective measure of attitude, but Brown and Abell (1965) found teacher observation to be inadequate as a method of appraising students' attitudes toward mathematics. On the other hand, Ellingson (1962) found a significant positive correlation ($r = .48$) between the inventoried mathematics attitudes of 755 junior and senior high school pupils and teachers' ratings of the pupils' attitudes.

Another fairly direct way of assessing attitudes is to ask the pupil how he feels about mathematics. This was the method used by Shapiro (1961) in a semi-structured interview of 19 questions aimed at determining the feelings toward arithmetic of 15 boys and 15 girls. The pupils' attitudes were determined by ratings of the 90 interviews made by three judges.

## Questionnaire Items

Dreger and Aiken (1957) administered the following three questionnaire items to a group of college students as a means of determining their anxiety or attitudes toward mathematics. The students responded to each item with true or false.

1. I am often nervous when I have to do arithmetic.
2. Many times when I see a math problem I just 'freeze up.'
3. I was never as good in math as in other subjects [p. 346].

More recently, Kane (1968) constructed another sort of questionnaire to measure attitudes toward mathematics and other school subjects. The college-student examinees were instructed to indicate which of four subjects--English, mathematics, science, and social studies--they most enjoyed and were most worthwhile in high school, they most enjoyed in college, they learned the most about in college courses, they would probably most enjoy teaching, and they were probably most competent to teach. Their attitudes toward mathematics were indicated by the extent to which mathematics was preferred or selected over the other three subjects. Other examples of non-scaled questionnaire items and more formal questionnaires such as those devised by the semantic differential technique (see Anttonen, 1967) could be supplied, but a more popular instrument for measuring attitudes is the attitude scale.

## Attitude Scales

There are several attitude-scaling procedures, a few of which will be described briefly. In Thurstone's method of successive intervals, each of a series of statements expressing different degrees of negative and positive attitudes toward something is given a scale value, the median of the scale values assigned to it by a group of judges. A respondent's score on a scale consisting of a series of such statements is the sum or mean of the scale values of the statements which he endorses.

In Likert's method of summated ratings, the respondent indicates whether he strongly agrees, agrees, is undecided, disagrees, or strongly disagrees with each of 20 or so statements expressing positive or negative attitudes toward something. His score on the scale is the sum of the weights (successive integers such as 1, 2, 3, 4, and 5) which have been assigned to the particular responses which he makes. On both the Thurstone and Likert scales, high scores indicate a more favorable attitude toward the particular thing.

The Thurstone and Likert attitude-scaling techniques are popular procedures for measuring attitudes toward mathematics, but a third method for scaling attitudes--Guttman's scalogram analysis--is employed less frequently. This is probably due to the fact that Guttman scaling requires a true scale, in the sense that if the respondent endorses one item he will endorse all items having a lower scale value. Such a restriction is more likely to be satisfied for cognitive test items than for non-cognitive items like attitude statements.

Example of a Thurstone scale. The scale of attitudes toward arithmetic which has probably been used more than any other is Dutton's scale (Dutton, 1951; 1962). This 15-item scale is given in Dutton's 1962 paper, and it consists of a variety of statements expressing positive and negative attitudes toward arithmetic. It was originally constructed to measure the attitudes of prospective elementary school teachers, but it has also been administered to junior high pupils (Dutton, 1968) and even as early as the third grade (Fedon, 1958). In Fedon's study (1958), the children indicated the intensity of their attitudes with a color scheme, varying from red for an extreme positive attitude through yellow to convey a neutral attitude and to black for an extreme negative attitude. Dutton's scale, like many others, is obviously multidimensional in that different statements assess attitudes toward different aspects of arithmetic.

Example of a Guttman scale. In an investigation to be discussed in more detail below, Anttonen (1967) arranged 94 attitude scale items into 15 Guttman-type scales. The obtained score was to be representative of the attitudes toward mathematics of fifth and sixth graders and of eleventh and twelfth graders, a rather wide range for any psychometric device.

Examples of Likert scales. Likert scales are usually easier to construct than Thurstone or Guttman scales, and therefore it is not surprising that many researchers have preferred this type of scale. In their book on attitude scales, Shaw and Wright (1967) have included two Likert scales for the measurement of attitudes toward mathematics--a 12-item, modified Likert scale by Gladstone, Deal and Drevdahl (1960), and Aiken's (1963) Revised Math Attitude Scale. The original version of the Aiken scale appears in an article by Aiken and Dreger (1961). Alpert, Stellwagon, and Becker (1963) described the quasi-Likert attitude scales used in the National Longitudinal Study of Mathematical Abilities (NLSMA) of the Stanford-based School Mathematics Study Group (SMSG). In the analysis of the NLSMA data, 40 attitude-type items were broken down into a number of subscales, for example, "pro-arithmetic composite," "actual arithmetic self-concept," and "debilitating anxiety." As a final example of a Likert scale, Dutton and Blum (1968) reworded the "strongest" items from Dutton's earlier Thurstone-type scale and formed them into a Likert-scale format.

### Other Measures of Attitude

Nealeigh (1967) experimented with a picture-preference test as a measure of pupil attitudes and achievement proneness in mathematics. The pupil was presented with 310 pairs of pictures, one member of each pair containing a mathematics concept, and told to indicate which of the two pictures he preferred. The "math concepts" included in the pictures were those of symmetry, similarity, order, and pattern. Attitude and achievement in mathematics were assessed in another way and compared with pupil responses to the picture-preference test. Although certain pictures discriminated between pupils with positive and negative attitudes and between pupils with high and low achievement, the pictures that were most discriminating with third graders were not necessarily the same ones that were most discriminating with seventh graders.

In the usual definition of the term, an attitude is viewed as partly cognitive and partly non-cognitive or emotional. Therefore, it would seem that information about attitude, or at least its emotional component, could be obtained by measuring autonomic responses to selected stimuli. Such measurements are too cumbersome for mass assessment of attitudes, but they have been employed in research. For example, Dreger and Aiken (1957) measured changes in electrical skin resistance (GSR) in 40 college students while the Verbal scale of the Wechsler-Bellevue Intelligence Scale (WB-I) was being administered to the students. Statistically significant GSR's were obtained during the arithmetic instructions and the arithmetic subtest of the WB-I, but only for those subjects who had been independently identified as anxious about mathematics.

Milliken and Spilka (1962) measured breathing depth, breathing rate, blood pressure, heart rate, and GSR during the first and last 30 seconds of the time that their subjects were taking each subtest of the American Council on Education Psychological Examination (ACE). The results showed that examinees who were low in mathematics and high in verbal score on the Scholastic Aptitude Test gave greater physiological responses during administration of the ACE mathematics tests. In addition, males in general gave greater physiological responses during the ACE verbal tests than during the mathematics tests, while the reverse was true for females.

## Grade Distribution and Stability of Attitudes

### The Elementary-School Years

It is generally recognized that attitudes toward mathematics in adults can be traced to childhood (Morrisett and Vinsonhaler, 1965, p. 132). There is evidence that very definite attitudes toward arithmetic may be formed as early as the third grade (Fedon, 1958; Stright, 1960), but these attitudes tend to be more positive than negative in elementary school (Stright, 1960). For example, a survey by Herman (1963) of the subjects least preferred by a group of fourth, fifth, and sixth graders found that arithmetic was typically in the middle when subjects were ranked from least to most preferred. For boys, the order of the five subjects, from least-liked to most-liked, was English, social studies, arithmetic, science, and spelling. For girls, the order, from least-liked to most-liked subject, was social studies, science, arithmetic, English, and spelling.

Indirect evidence of the grade distribution of mathematics attitudes is found in reports given by groups of college students majoring in education. The students stated that they developed their attitudes toward arithmetic throughout the school grades--from second through twelfth grade--but that the intermediate grades--fourth through sixth--were more influential (Dutton, 1962; Smith, 1964; White, 1963). This seems reasonable, because these are typically the three grades in which arithmetic is stressed most. In McDermott's (1956) case studies of 34 college students who were afraid of mathematics, the majority reported having first met with frustration in the elementary grades; the remainder stated that they met difficulty when they attempted the use of algebraic symbols and other higher math concepts in secondary school.

Interestingly enough, there is some evidence of a decline from the third through the sixth grade in the percentage of pupils expressing negative attitudes toward arithmetic (Stright, 1960). However, the change may be due to increasing social sophistication on the part of the pupils, or an increased willingness to simulate positive attitudes because they have been told that mathematics is good for them and positive attitudes please the teacher.

## The Junior-High-School Years

The results of a number of studies point to the persistence of negative attitudes toward mathematics as students ascend the academic ladder. In the traditional curriculum the junior high school has been the period during which algebra and other abstract mathematics were introduced, and this is the time during which many of the writer's friends have stated that they began to dislike the subject. It is noteworthy that the greatest percentage (40%) of the prospective teachers surveyed by Reys and Delon (1968) listed the junior-high-school years as the period when their attitudes toward arithmetic developed. And even under more contemporary modern mathematics curricula, junior high school seems to be a critical period in the determination of attitudes toward mathematics (Dutton, 1968).

Attitudes toward new math. Dutton and Blum (1968) made a survey of the reasons for disliking and liking arithmetic in 346 sixth-, seventh-, and eighth-grade pupils who had been taught "new math" for at least one year. The most frequent reasons for disliking the subject were: working problems outside of school, word problems that were frustrating, possibilities of making mistakes in arithmetic, and too many rules to learn. A large percentage of the pupils agreed with the statements that arithmetic should be avoided whenever possible, that one cannot use new mathematics in everyday life, and that arithmetic is a waste of time. Favorable attitudes expressed by pupils were that working with numbers is fun and presents a challenge, and that arithmetic makes you think, is logical, and practical.

Dutton (1968) suggests that there has been a decline during the past ten years in the number of junior-high-school pupils expressing negative attitudes toward arithmetic, but that a sizable percentage of pupils are still not sure of themselves in the subject. The reviewer notes, however, that Dutton's procedure was to compare the attitudes of a group of junior-high-school pupils of a decade ago with those of a current group, a procedure which undoubtedly did not result in equivalent groups.

## Longitudinal Studies of Mathematics Attitudes

Obviously, what we need in order to assess the grade distribution and stability of attitudes toward mathematics are both cross-sectional and longitudinal surveys. But one difficulty in obtaining this information is the possible inappropriateness of the same attitude measure at different grade levels. Fedon's (1958) use of a color scheme for indicating intensity of attitude in the lower grades represents an interesting attempt to extend downward a scale which was constructed for students on a much higher level.

Actually, there have been very few longitudinal studies of attitudes. An unpublished analysis by the writer of the mean scores on the SMSG mathematics attitude scales obtained by the same group of approximately 1000 children in grades four, six, and eight revealed significant changes across grade levels in mean scores on some scales, although these were not very dramatic (data from Wilson et al., 1968). Anttonen (1967) administered 94 attitude items, arranged into 15 Guttman-type scales, to 607 fifth- and sixth-grade Minnesota school children in 1960. The scales were readministered to a portion of the same group six years later, when they were in the eleventh and twelfth grades, respectively. The correlation between mathematics attitudes in elementary and secondary school was relatively low (average $r$ of .30) for the entire group, for grades and sexes considered separately, and for four different patterns of mathematics coursework. However, scores on the Guttman attitude scale administered in senior high had a high correlation with a semantic differential measure of attitudes administered during the same period.

In sum, it seems possible to measure attitudes toward arithmetic or mathematics as early as the third grade, but, as in any interest pattern affected by development, such attitudes are probably not very stable in the early grades. In addition, the preciseness with which pupils can express their attitudes varies with level of maturity. Finally, it is clear that attitudes toward different aspects of arithmetic and mathematics are measured by "general attitude" instruments administered at different grade levels. Attitude toward materials to be learned by rote, such as the multiplication table, is not the same variable as attitude toward word problems and algebraic symbols. Of more importance than the exact frequency of attitudes at different grade levels, however, are the causes and effects of these attitudes.

## The Relationship of Attitude to Achievement in Mathematics

Obviously, the assessment of attitudes toward mathematics would be of less concern if attitudes were not thought to affect performance in some way. But assuming that attitudes do affect performance, what are the dynamics by which this is thought to occur? Bernstein (1964) maintained that if certain feelings are experienced for a time they will lead to a particular self-image by the pupil--a self-image which will influence his expectation of future performance, with consequent effects on actual performance. Data collected by Kempler (1962), and bearing on this assertion, suggest that self-confidence in mathematical ability, as measured by a 15-item questionnaire, is associated with rigidity in mathematical tasks such as the Luchins water-jar test. Behaviors indicative of the rigidity which students manifest toward frustrating mathematical tasks, which cause them to be anxious and hostile toward the subject, are resorting to rote and inefficient methods, and relying on other people and dishonest means in order to pass (McDermott, 1956). In contrast to the rigidity and "giving up" observed in those who dislike mathematics is the constructive perseverance of those who like mathematics. Shapiro (1961) found that perseverance toward solutions to arithmetic problems was higher in elementary-school children who liked mathematics than in those who disliked it, girls as a group being more persevering than boys.

A similar analysis of the relationships among attitude, expectation, and performance was made by Alpert et al. (1963), who view level of expectation and performance as a kind of self-perpetuating cycle affecting a child's self-concept; attitudes and anxiety are closely related to this concept. The notion of a self-perpetuating cycle linking expectation and performance is consistent with the observation that the variability of arithmetic performance increases as pupils proceed through elementary school. That is, the difference between the poorest and the best pupils becomes progressively greater (Clark, 1961).

The relationship of attitudes, (which are integrally related to expectations), to performance appears to be especially important in mathematics. As an illustration, the results of one study (Brown and Abell, 1965) were that the correlation between pupil attitude and achievement was higher for arithmetic than for spelling, reading, or language. But let us examine, by grade level, the results of research over the past decade on the relationship between attitudes and achievement in mathematics.

## Elementary-School Level

In a study of the attitudes toward problem solving in a group of Brazilian elementary-school children, Lindgren et al. (1964) obtained a small but significant positive correlation between problem-solving attitudes (Carey, 1958) and arithmetic achievement, and a positive but not significant correlation between attitudes and marks in arithmetic. Shapiro (1961) found her interview measure of attitudes in sixth graders to be significantly related to grade placement on the Wide Range Achievement Test, all parts of the arithmetic section of the California Achievement Test, and to school marks in arithmetic. In another study, Anttonen (1967) obtained consistently low correlations of math attitude scores with grade averages and with the arithmetic total scores on the Iowa Tests of Basic Skills of fifth- and sixth-grade pupils. Indirect evidence for a relationship between attitude and achievement comes from a survey by Dutton (1962), who found a low positive correlation between the attitudes toward arithmetic in college students and their reported arithmetic grades in elementary school.

Quite obviously, the correlations between attitude and achievement in elementary school, although statistically significant, are typically not very large. In fact, one investigation of sixth graders (Cleveland, 1961) found that attitude scale scores did not generally discriminate between high and low achievers in arithmetic. One difficulty with self-report inventories at the elementary-school level is the readability and interpretability of the attitude instrument; another is the self-insight and conscientiousness with which the pupils fill out the inventory. Hopefully, these problems are not so serious at higher grade levels.

## Junior-High-School Level

Summarizing the results of a survey of 270 seventh-grade boys and girls, Alpert et al. (1963) reported significant correlations between performance in mathematics and measures of attitudes and anxiety toward mathematics. Similar results are given by Degnan (1967), Stephens (1960), and Werdelin (1966). In a comparison of accelerated and remedial mathematics classes, Stephens (1960) administered Dutton's attitude scale to six seventh-grade and six eighth-grade classes. The mean attitude score of the accelerated group was significantly higher than that of the remedial group. Therefore, Stephens concluded that attitude scores might be used, together with achievement test scores, for placement in special classes.

Degnan (1967) compared the attitudes and general anxiety levels of 22 eighth-grade students designated as low achievers in mathematics with those of 22 eighth-grade students designated as high achievers in mathematics. Dutton's scale was the measure of attitudes, and Casteneda's Manifest Anxiety Scale the measure of general anxiety. Although it was found that the achievers were generally more anxious than the underachievers, the achievers had much more positive attitudes toward mathematics. Also, when the students were asked to list their major subjects in order of preference, the achievers gave mathematics a significantly higher ranking than the underachievers. Among other things, the results of this study show that attitude toward arithmetic and general anxiety are not the same variable, a conclusion related to the earlier finding of Dreger and Aiken (1957) that "general anxiety" and "math anxiety" are not the same. The study also demonstrates that anxiety may act as a facilitating factor in achievement, as noted by Alpert et al. (1963) in the NLSMA variable of "facilitating anxiety" in mathematics.

## High-School Level

In his longitudinal study of attitudes, Anttonen (1967) reported moderate correlations of mathematics attitude scores with mathematics grade-point averages and standardized test scores in eleventh and twelfth graders. Achievement was also greater for students whose attitudes had remained favorable or had become favorable since elementary school.

## College Level

Due perhaps to the greater accessibility of subjects, it is not surprising that many investigators prefer to work with college students. Since college students, on the average, presumably have more positive attitudes toward academic work than their non-college counterparts, it would seem that the frequency of negative attitudes toward mathematics, and consequently the variability of the distribution of attitude scores, should be lower for college students than for the general population. If this is true, then one might expect a somewhat smaller correlation between attitudes and achievement in college than in high school. On the other hand, college students may fill out attitude inventories more conscientiously and with greater self-insight than the population as a whole--factors promoting higher attitude-achievement correlations.

Some investigators have found rather low correlations between mathematics attitudes and mathematics achievement in college students. For example, Harrington (1960) reported a statistically insignificant relationship between attitude and performance in college mathematics courses, although he did find that selection of a mathematics course vs. no mathematics course was significantly related to attitude. Somewhat more substantial relationships between attitudes and achievement were obtained by Dreger and Aiken (1957) and Aiken and Dreger (1961). In the former study, there was a correlation of -.44 between the final grades of 704 students in a freshman mathematics course and their scores on a three-item inventory of anxiety in the presence of mathematics. In the second study (Aiken and Dreger, 1961), scores on the Math Attitude Scale contributed significantly to the prediction of the final mathematics grades of 67 college women when combined in a regression equation with high school mathematics averages and scores on the Verbal Reasoning and Numerical Ability tests of the Differential Aptitude Tests. However, the Math Attitude Scale was not a significant predictor for the 60 college men. Finally, there were statistically significant part correlation coefficients for both males ($\underline{r}$ = .33) and females ($\underline{r}$ = .34) between Math Attitude Scale scores and scores on a retest of the Cooperative Mathematics Pretest for College Students, after initial scores on the latter variable had been partialed out.

## Attitude as a Moderator Variable

The Aiken and Dreger (1961) study is an illustration of the multiple correlation approach to prediction, in which measures of attitude and ability were combined in a regression equation to predict achievement. A second prediction approach is to view attitude as a moderator variable and to determine the correlation between ability and achievement separately at each of several levels of attitude. Thus, it may be discovered that the correlation between ability and achievement varies with attitude.

Cristantiello's study (1962) is an example of this moderator variable approach. College sophomore men (N = 264) were classified by area of major (business administration, social science, natural science), and within each of these areas further divided into three levels (high, middle, low) according to their scores on a scale of attitude toward mathematics. Then the correlation between scores on a measure of quantitative ability (ACE-Q scores) and mathematics grades was found separately for each of the nine major area × attitude level groups. The correlations between ACE-Q scores and mathematics grades were significantly more positive for students with middle attitude scores, and significantly lower for those with low attitude scores. These results could not be explained by differences among the groups in variances of either grades or ACE-Q scores.

Although Cristantiello's results could bear replication, they may be interpreted as indicating that mathematical ability may be a less important determiner of the achievement of students having more extreme attitudes toward mathematics than of those having more moderate attitudes. Related to these findings is Jackson's (1968) conclusion that attitude scores in the middle range of values have little relation to achievement. He maintains that it is only at the extremes--highly positive or highly negative--that attitude affects achievement in any significant way. If Jackson is correct, then it is reasonable to expect that in the middle range of attitude scores, as was found by Cristantiello, ability scores rather than attitude scores will be more accurate predictors or determiners of achievement.

## An International Study of Attitudes and Achievement

In an international study designed to compare the mathematics achievement of 13- and 17-year-old (terminal secondary) students in a dozen countries (Husén, 1967), extensive data concerning attitudes, interests, and certain other variables were also collected. Three of the five attitude scales which were administered were described as measures of attitudes toward mathematics as a process, attitudes about the difficulties of learning mathematics, and attitudes about the place of mathematics in society. One of the findings concerning scores on the first scale--a measure of the extent to which mathematics is viewed as fixed, as opposed to developing or changing--was that in all countries studied the upper-level (older) students considered mathematics as less changing than did the lower-level (younger) students. There was also a tendency for students in countries in which the "New Mathematics" was taught to see mathematics as more open and changing.

With respect to scores on the second scale--a measure of the perceived difficulty of learning mathematics--upper-level students tended to perceive mathematics as more difficult and demanding. Interestingly enough, scores on the third scale--a measure of the perceived role of mathematics in contemporary society--indicated that mathematics was viewed as less socially vital or valuable by students with the longest exposure to it and by students in countries where English is spoken.

Some of the correlational results of this international investigation were: significant negative rank-order correlations between mean mathematics achievement and mean scores across countries on the attitude scales; rather small correlations between achievement and attitude within countries; moderate to high correlations between achievement and interest measures within countries. In summarizing the results referred to above, the author (Husén, 1967) concluded: "We may say, in general, that in those countries where achievement is high pupils have a greater tendency to perceive mathematics as a fixed and closed system, as difficult to learn and for an intellectual elite, and as important to the future of human society" [p. 45].

## Relationship of Attitudes to Personality and Social Factors

### Anxiety and Attitude

As was noted above, an attitude is emotional as well as cognitive, so some relationship between a measure of attitude and a measure of anxiety toward a particular school subject should be expected. In addition, anxiety and attitude may be either general or specific, pertaining to only one situation or event or to many. In this regard, a number of studies during the past decade have related scores on Casteneda's Manifest Anxiety Scale (CMAS)--presumably a measure of debilitating anxiety--to performance in mathematics (e.g., McGowan, 1960; Reese, 1961). Typically, these studies have found small but statistically significant negative correlations between manifest anxiety and achievement, correlations usually somewhat lower than those between attitude and achievement in a specific subject. Thus, Reese (1961) obtained a correlation of -.25 between CMAS scores and the arithmetic achievement of fourth- and sixth-grade girls, when IQ was partialled out.

*General and specific attitudes*. The relationship between attitude toward academic work in general and attitude toward mathematics in particular has also been investigated, although here there is an apparent inconsistency in the findings of two investigations. In a study conducted in Sweden, Werdelin (1966) administered a questionnaire concerning attitudes toward school work and mathematics to ninth graders. A close relationship between attitudes toward school work in general and attitudes toward mathematics was reported. This finding contrasts with that of Aiken and Dreger (1961): a test of independence between scores of college students on the Math Attitude Scale and their scores on four items designed to measure attitudes toward school work in general was not significant. There are several possible explanations for the difference between the findings of the two investigations: age level, nationality, and the measuring instruments were not equivalent in the two samples. Nevertheless, it is possible to construct an inventory to measure anxiety or attitude (Aiken and Dreger, 1961) which is fairly specific to mathematics.

## Intellective Factors

Although it has been observed that general ability to learn is associated with liking for arithmetic (see Brown and Abell, 1965), typically, have rather low measures of anxiety and attitudes toward school subjects, have rather low correlations with measures of intellectual ability (Aiken, 1963; Dreger and Aiken, 1957; Lindgren et al., 1964). Dreger and Aiken (1957) found, for example, that reported anxiety in the presence of mathematics had a statistically insignificant correlation of -.25 with ACE Quantitative scores and a correlation of only -.08 with ACE Linguistic scores. Lindgren et al. (1964) found near-zero correlations between Carey's (1958) measure of problem-solving attitude and intelligence test scores in a group of fourth-grade pupils in Brazil. For a group of 160 college women, Aiken (1963) obtained an insignificant correlation between Math Attitude Scale scores and Scholastic Aptitude Test (SAT) Verbal scores, but attitude scores were significantly correlated with SAT Quantitative scores ($r = .37$).

Two comments concerning these data may be made. Actually, one might expect attitude toward a specific subject to be significantly related to a measure of ability in that subject. This is because measures of specific ability and specific achievement in a given area are closely associated, and achievement affects attitude and vice versa. In addition, the significant correlation of .37 between attitude and ability for the 160 women students in the Aiken (1963) study is consistent with unpublished data collected by the writer demonstrating that attitude scale scores are more highly correlated with both ability and achievement measures in the case of females than in males. The reader will also recall the results of an investigation summarized above (Aiken and Dreger, 1961) in which attitude was a significant predictor of mathematics achievement for females but not for males. The factor of sex differences in attitudes toward mathematics will be discussed in more detail below.

## Social Factors

One possible social determiner of attitude toward mathematics is the attitudes of one's peers. Shapiro's (1961) findings indicate that peer attitudes in elementary school may indeed be influential, especially in the case of girls. The social influences of parents and teachers will be treated in detail in later sections of this review. Otherwise, the effects of social factors on attitudes toward mathematics appear to be relatively unimportant. The fact that negative math attitudes are not produced by only one type of school system is documented by McDermott (1956), who found that the backgrounds of students who were afraid of mathematics ranged from one-room rural schools to large city school systems. Alpert et al. (1963) also referred to measures of parental socioeconomic status as being included in their investigation, but apparently they were not related to mathematics attitudes in any significant way. Lindgren et al. (1964) reported an essentially zero correlation between socioeconomic status and Carey's (1958) measure of problem-solving attitudes, and Hungerman (1967) obtained negligible correlations between socioeconomic status and mathematics attitudes in a group of sixth graders.

Although there is some evidence that higher mathematics achievement goes with a higher socioeconomic environment (Cleveland, 1961), mathematics test scores are usually not as highly related as verbal test scores to socioeconomic status (Karas, 1964). Karas (1954) maintained that the home environment has a greater effect on performance in more verbal subjects than in subjects such as mathematics that are more highly loaded with less familiar symbolic material. Considering the positive relationship between attitude and achievement, one may generalize from Karas' findings that socioeconomic status and perhaps other home factors have less effect on attitude toward mathematics than on attitude toward more verbal subjects. The writer is not aware of any research specifically designed to test this hypothesis, but a number of studies have been concerned with the relationships of parental attitudes and encouragement to student attitudes toward mathematics.

## Parental Influences

According to Poffenberger and Norton (1959), parents affect the child's attitude and performance in three ways: (a) by parental expectations of child's achievement; (b) by parental encouragement; (c) by parents' own attitudes. As evidence for their hypothesis that the conditioning of children's attitudes occurs in the family, they cite the results of a study of 390 University of California freshmen. The students filled out a questionnaire concerning their own attitudes and the attitudes and expectations of their parents. The findings were that the students' attitudes toward mathematics were positively related to how they rated their fathers' attitudes toward mathematics. The attitudes of the students were also related to their reports of the level of achievement in mathematics which their fathers and mothers expected of them. Poffenberger and Norton (1959) suggested that attitudes reported for mothers were not significantly related to students' own attitudes because only a small number of students indicated that their mothers liked mathematics.

In a further analysis of self-report data, Poffenberger (1959) found that college students who reported a distant relationship with their fathers showed a significant tendency to perceive their fathers as disliking mathematics. In contrast, students who reported a close relationship with their fathers did not differ from the total sample of students in their ratings of their fathers' attitudes toward math. However, Poffenberger did not interpret these data as offering support for the hypothesis that attitude toward mathematics is caused by the warmth of a child's relationship with his father--the masculine identification model. Rather, the results were seen as being due to a generalized perception on the part of students, viz.: children who feel that their parents do not like them (since they are not close to them) perceive the parents as negatively oriented to other aspects of life as well, for example, mathematics. The relationship between masculine identification and attitude toward mathematics will be treated in more detail below, but several other studies concerned with parental attitudes and expectations should be reviewed first.

Aiken and Dreger (1961) found no significant relationships between Math Attitude Scale scores and student reports on the degree to which parents emphasized and encouraged school work when the students were children. It is noteworthy that although none of the correlations for male or female students was significantly greater than zero, the correlations for females were uniformly more positive than for males.

The three studies reviewed above were concerned with student reports of the expectations and attitudes of their parents. More direct information on the relationships of student attitudes to parental expectations and attitudes was obtained by Alpert et al. (1963) and Hill (1967). Alpert et al. (1963) developed a parental interview and questionnaire to determine the extent to which parental attitudes and values are consistent with those of the School Mathematics Study Group and affect the attitudes of their seventh-grade children toward mathematics. These were the results: (a) student attitudes, for both boys and girls, were positively correlated with the amount of mathematics education desired by parents for their children; (b) boys' attitudes were positively correlated with the importance which their parents placed on grades and with parental demands for higher grades, whereas girls' attitudes toward mathematics were negatively related to the importance that their parents placed on mathematics; (c) student attitudes for both boys and girls were positively correlated with parents' views of competition as necessary in the modern world and as good. An interesting sex difference also occurred with respect to these parent variables. Parents of boys who had positive mathematics attitudes tended to view the goal of a junior-high mathematics program as "to aid the intellectual development of the child"; parents of girls who had positive mathematics attitudes tended to see the goal of a junior-high mathematics program as "ability to deal competitively with practical everyday problems." Conversely, the parents of boys with negative mathematics attitudes saw the goal of a junior-high mathematics program as "ability to deal competitively with practical everyday problems," whereas the parents of girls with negative mathematics attitudes tended to view the goal as "to aid the intellectual development of the child."

Hill (1967) interviewed the fathers and mothers of 35 upper-middle-class boys and administered a questionnaire concerned with attitudes toward mathematics to their sons. He found a greater similarity between the attitudes of mothers and sons than between the attitudes of fathers and sons. The degree of similarity in attitudes between mothers and sons was related to maternal warmth, use of psychological control techniques, and low paternal participation in child rearing. Parental attitudes and expectations for their sons were not significantly related, but sons did show greater accordance with the expectations of their fathers than with those of their mothers. The variables of father warmth and degree of participation in child rearing were positively related to degree of sons' accordance with fathers' expectations. Also, fathers who had greater expectations of masculine behavior in their sons and who viewed mathematics as a masculine subject had a higher level of aspiration in mathematics for their sons. Quite obviously, Hill's (1967) data cannot be handled adequately by the theory that positive attitudes toward mathematics are due to masculine identification. But we need to look a little further into the data on sex differences and masculinity vs. femininity of interest before drawing conclusions about the adequacy of any sex-identification hypothesis.

## Sex Differences

No one would deny that sex can be an important moderator variable in the prediction of achievement from measures of attitudes and anxiety. The results of several of the investigations discussed so far (e.g., Aiken and Dreger, 1961; Reese, 1961) have suggested that measures of attitudes and anxiety may be better predictors of the achievement of females than of males. Traditionally, mathematics has been viewed as more of a man's interest or occupation, and consequently one might expect that males would score higher than females on tests of ability and achievement in mathematics and on scales of attitudes toward mathematics. Norms on the mathematics sections of tests like the Differential Aptitude Tests and the Scholastic Aptitude Tests do indicate higher mean scores for males than for females at the high school level, a sex difference which has been interpreted as being produced by greater cultural reinforcement of interest and pursuit of mathematics in males at the higher grade levels. Although boys have traditionally been viewed as better than girls in problem solving (see Sweeney, 1954), one recent study of eleventh graders (Meyer and Bendig, 1961) found a superiority on the part of girls in the number and reasoning factors of the Primary Mental Abilities Test. Two recent studies of sex differences in arithmetic at the elementary-school level found no difference between the performance of boys and girls or a superiority on the part of girls, depending on the test and the grade level (Shapiro, 1961; Wozencraft, 1963).

More specific to sex differences in attitudes toward mathematics are Stright's finding (1960) that elementary-school girls liked arithmetic better than the boys, and Dutton's (1968) finding that girls and boys who had studied "new math" were about equal in their liking for arithmetic. On the other hand, in studies at the college level (Aiken and Dreger, 1961; Dreger and Aiken, 1957), the reviewer has consistently found a significantly more positive mean attitude toward mathematics in males.[2] Assuming equivalent samples, the difference between the results at the lower grade levels and at the college level may be due, as was noted above, to differential cultural reinforcement for males in mathematical endeavors, beginning at the secondary school level. In addition, any explanation of the discrepancy in results must take into account interactions between the sex variable and accuracy of attitude measures in the earlier school grades, desire to please the teacher, and rate of academic maturation in general.

## Masculinity-Femininity of Interest

A not uncommon finding concerning the interest patterns of those who like and dislike mathematics is that reported by McDermott (1956) in a case-study comparison of 34 college students who feared math with seven students who were proficient in the subject. McDermott found that those who had developed a fear of mathematics preferred English, social studies, and the arts, but disliked the definiteness of mathematics. The students who were proficient in mathematics were critical of the vagueness of the humanities and were not interested in majoring in the area. A hypothesis related to McDermott's findings and referred to in Feierabend's earlier review (1960) is that interest and ability in mathematics are a consequence of masculine identification. What research tests of this hypothesis have there been since 1960?

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[2] In an assessment of the attitudes toward mathematics of 264 Fairleigh Dickinson University students, Roberts (1969) reported no significant sex differences in attitudes, but engineering students held more positive attitudes than students in terminal mathematics programs.

As a test of the above hypothesis, Lambert (1960) administered the ACE, an arithmetic skills test, and the MMPI to 1372 U.C.L.A. undergraduates. Group I consisted of 80 students in advanced mathematics or physics courses, and Group II was composed of 1292 senior education students. Contrary to the masculine-identification hypothesis of Plank and Plank (1954), Lambert found no correlation between mathematical proficiency and MMPI Masculinity-Femininity (Mf) scores in either sex in any of the groups. In addition, the mean Mf score of the 10 female mathematics majors was significantly more feminine than that of the 744 female education majors. Finally, there was no significant difference between the mean scores of the male mathematics majors and the male education majors on the Mf scale of the MMPI. As a comment on this investigation, the comparatively small number of female mathematics majors casts some doubt on the generalizability of the results. In addition, the factor of general intelligence was not controlled and may have affected the results. For example, the selected group of mathematics majors may have been more intelligent and, therefore, perhaps more interested in cultural (i.e., "feminine") pursuits than typical persons with positive attitudes toward mathematics. Also, it is uncertain how representative the group of education majors was of the general college population; a comparison group randomly selected from all major fields should have been chosen. Finally, the MMPI Mf scale is not necessarily the best measure of masculinity-femininity of interests. In any event a study like Lambert's (1960) is fairly easy to carry out, and it should be replicated and extended, in light of the criticisms made above, at other schools and colleges.

In another test of the masculine-identification hypothesis at the college level, Carlsmith (1964) obtained student reports of the length of time that their fathers had been absent from home when the students were children. These time reports were compared to the students' scores on the Verbal and Mathematics sections of the Scholastic Aptitude Test (SAT) and to the difference between SAT-Verbal and SAT-Mathematical scores. The results were that, for both boys and girls, the longer the father was absent from the child during early childhood, the lower was the latter's math score relative to his verbal score. An additional finding was that if the father was absent for a short period of time during a boy's adolescence, the boy's mathematics score was higher than in cases where the father was not absent at this time. As a explanation of these results, Carlsmith dismissed the hypothesis that separation from the father produces anxiety and anxiety affects mathematics scores more than verbal scores. He maintained that the masculine conceptual approach, which is necessary to achieve in mathematics, is acquired through close and harmonious association with the father. Certainly Carlsmith's investigation, like that of Lambert (1960), bears replicating, because there is an apparent disagreement between the results of the two studies. Again, however, one must be cautious about the method used to measure masculine identification. As a way of linking the results of the Carlsmith and Lambert studies, it may be of interest to determine the relationship between father absence during early childhood and scores on a masculinity-femininity interest measure such as the MMPI Mf scale.[3]

The purpose of a study by Elton and Rose (1967) was to test the hypothesis that girls avoid mathematics because they view it as a masculine activity. It was predicted that girls who had high scores on the English section but only average scores on the Math section of the American College Test (ACT) would show more feminine interests on the Omnibus Personality Inventory (OPI). In contrast, girls with average scores on the ACT English section and high scores on the ACT Math section should manifest more masculine interests on the OPI. The scores on the ACT and OPI of females in the 1962-1965 classes at the University of Kentucky were analyzed. Students' scores were classified as low, average, and high on the ACT mathematics and English tests, and the data on students showing seven of the nine possible combinations (e.g., high in English and low in mathematics or low in English and average in mathematics) were related by multiple discriminant analysis to the factor scores on the 16 OPI scales. The results indicated that girls in the high English-average mathematics group were more interested in cultural and artistic (i.e., more feminine) matters, whereas girls in the average English-high mathematics group had more theoretical and fewer esthetic (i.e., more masculine) interests. The difference between masculinity-feminity of interest was also in the predicted direction for the low English-average mathematics and average English-low mathematics groups, the former group showing more masculine interests on the OPI than the latter group. Thus, as in the Carlsmith (1964) study, masculine identification, or masculine role, was a predictor of large differences between verbal and mathematics scores on a college entrance test. The findings of Elton and Rose (1967), however, are perhaps more easily accepted and interpreted than those of Carlsmith, because they do not require that we reach back to an event in a person's early childhood as an explanation of the difference between his verbal and mathematics scores on a college admissions test. Many of the items on the OPI concern interests in reading, science, and other verbal-related and mathematics-related pursuits. And it is not surprising that girls with more verbal-related (viz., "cultural") interests, as measured by the OPI, should also have higher verbal-ability than mathematics-ability scores, whereas girls with more mathematics-related (viz., scientific, theoretical) interests should have higher mathematics-ability than verbal-ability scores. It is not necessary to argue whether the scientific-theoretical (masculine) interest or the mathematics ability came first, or whether the cultural (feminine) interest or the verbal ability came first. The two factors--interest and ability--form a mutually reinforcing system. There are obviously many other factors that enter into the equation, but in general people tend to like those things which they do well, and, perhaps to a more limited extent, they tend to do well in those things they like.

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[3] Another interesting hypothesis of Carlsmith (1964) is that aptitude for mathematics is fairly well established by the fourth grade and highly resistant to change during subsequent years. But mathematics aptitude is certainly not a unitary factor, and the different mathematical abilities presumably mature at different rates and are differentially affected by experience. A thorough longitudinal study is needed to trace the growth of various mathematical abilities from preschool onward, assuming that appropriate tests of such abilities can be constructed.

Sex role is only one of the personality variables which are related to attitude toward and performance in mathematics. Obviously, there are many sources of within-sex differences in attitudes. These differences in attitudes are certainly related to differences in ability, but they may also be related to other personality variables. In addition, it may be of interest to review some of the investigations concerned with the relationships between achievement in mathematics and personality variables other than sex role, since the results may shed some light on the dynamics of attitudes toward mathematics.

### Other Personality Variables

Correlations with attitudes. In an initial study employing the Math Attitude Scale, Aiken and Dreger (1961) found little relationship between mathematics and scores on the seven scales of the Minnesota Counseling Inventory (MCI). The MCI Leadership scale had the highest correlation with Math Attitude scores for the 60 college men ($r = -.21$), and there was a low but significant positive relationship between the Math Attitude Scale scores and the MCI Adjustment to Reality scale scores of 67 college women. More evidence of the relationships of mathematics attitudes to a broad constellation of personality variables was obtained by Aiken (1963). For 160 college women, scores on the

Revised Math Attitude Scale were significantly correlated with 15 out of a total of 40 scales on three personality inventories--the California Psychological Inventory (CPI), the Sixteen Personality Factor Questionnaire (16 PFQ), and the Allport-Vernon-Lindzey Study of Values (SV). When scores on the SAT-Mathematical test were partialled out, six of the 15 correlations were still statistically significant--the correlations between mathematics attitudes and CPI Dominance, CPI Self-Control, CPI Achievement via Conformance, CPI Intellectual Efficiency, 16 PFQ Integration, and SV Theoretical Scale. Aiken (1963) interpreted these results as demonstrating that high scorers on the Revised Math Attitude Scale, with mathematical ability controlled, tend to be more socially and intellectually mature, more self-controlled, and to have more theoretical interests than low scorers on the scale.

Correlations with achievement. Feierabend (1960, pp. 21-23) devoted three pages of her review to research relating personality variables to achievement in mathematics. Since achievement and attitude are related, it may be worthwhile to summarize briefly the results of two studies on the topic which have been completed since 1960. In a study of sixth-grade pupils, Cleveland (1961) divided the group into three IQ ranges: 75-89, 90-110, and 111-125. Although scores on the California Test of Personality (CTP) did not significantly discriminate between low achievers and high achievers in mathematics among children in the 75-89 and 111-125 IQ ranges, there were several significant differences in personality test scores between low and high achievers in the 90-110 IQ range. High achievers in the 90-110 IQ range had significantly higher scores than low achievers on CTP Sense of Personal Worth, Sense of Personal Freedom, and Community Relations. The investigators interpreted the lack of significant differences in personality between low and high achievers in the 75-89 and 111-125 IQ ranges as being due to the greater influence of intellective factors in these ranges.

Collectively, the findings of studies relating personality variable to mathematics attitudes and mathematics achievement indicate that individuals with more positive attitudes and higher achievement tend to have better personal and social adjustment than those with negative attitudes and low achievement. These results must be kept in perspective, however. The correlations are relatively low, and it is a truism that correlation does not imply causation. Personal-social adjustment, attitudes, and achievement not only interact with each other, but they are the effects of other home, school, and community variables. Recent research on the home variables has already been examined, so we now turn to the school and especially to the teachers of arithmetic and mathematics.

## Teacher Characteristics, Attitudes, and Behavior

It is rather generally held that teacher attitude and effectiveness in a particular subject are important determiners of student attitudes and performance in that subject. As an example of research bearing on this supposition, Torrance et al. (1966) studied 127 sixth- through twelfth-grade mathematics teachers who participated in an experimental program to evaluate SMSG instructional materials. Pre- and posttests of educational and mathematical progress, aptitude, and attitude were administered, with the result that teacher effectiveness had a positive effect on student attitudes toward teachers, methods, and overall school climate.

It is also true of course that students who do not do well in a subject may develop negative attitudes toward that subject and blame their teachers for their failures, even when the teachers have been conscientious. Thus, it is possible to interpret the findings of Aiken and Dreger (1961) as being due as much to "sour grapes" on the part of the students as to objective characteristics of their mathematics teachers. A result of this investigation was that college men who disliked mathematics, as contrasted with those who liked mathematics, stated that their previous mathematics teachers had been more impatient and hostile. College women who disliked mathematics, in contrast to those who liked mathematics, tended to view their previous mathematics teachers as more impatient, not caring, grim, brutal, dull, severely lacking in knowledge of the subject, and not knowing anything about how to teach mathematics. In many of the correlational studies to be reviewed below, there will be a similar problem of deciding which variable is cause and which effect, or, as was discussed above, whether the two variables form a mutually reinforcing system. In spite of the difficulty of making clear interpretations, the results of these investigations may stimulate more controlled research on the topic.

### Interactions Between Teacher Attitudes and Student Attitudes

Garner (1963) administered an inventory of attitudes toward algebra to 45 first-year algebra teachers and their 873 Anglo-American and 290 Latin-American pupils in a Texas school system at the beginning and end of the school year. Standings in beginning attitudes, in judgments concerning the practical value of algebra, and in algebra achievement were significantly higher in Anglo-American than in Latin-American pupils. Significant relations were found between: (a) teacher's background in mathematics and student achievement in algebra; (b) teacher's attitude toward algebra and students' attitudes; (c) teacher's and students' judgments concerning the practical value of algebra; (d) teacher's attitude and changes in attitudes toward algebra in the Latin-American students.

Peskin (1964) studied the relationship of teacher attitude and understanding of seventh-grade mathematics to the attitudes and understanding of students in nine New York City junior high schools. Correlations were computed between the scores of teachers and students on six tests of attitude toward and understanding of arithmetic and geometry. The correlations between teachers' and students' understanding of algebra and geometry were significantly positive, as were the correlations between teachers' understanding scores and students' attitudes. The relationships of teacher understanding and attitude to student achievement and attitude were complex. For students having very high or very low levels of achievement, the correlations between teacher understanding and student achievement were significantly positive in the cases of both arithmetic and geometry. On the other hand, the correlation between teacher understanding and student attitude was significantly negative for the very high level group in geometry. There was also an interaction between teacher attitude and understanding, in that teachers with a "middle" attitude and a "high" understanding had students with the best scores in geometry, but teachers with "high" understanding and "low" attitudes had students with the poorest achievement in arithmetic and geometry.

*Cross-lagged panel correlation*. These results pose again the "chicken-egg" or cause-effect question referred to above. In short, do teacher attitudes and achievements affect student attitudes and achievements or vice versa? Simple correlation analysis cannot answer this question, but there is a correlational procedure which may give some information on which source--the pupil or the teacher--has the greater effect on the other's attitude and achievement. Campbell and Stanley (1963, pp. 68-70) have discussed such a design involving time as a third variable, which they refer to as "cross-lagged panel correlation." As an illustration of the approach, suppose that an attitude scale is administered to a group of teachers and their students at time 1 (pretest) and readministered at time 2 (posttest). Then the correlation between teachers' attitudes at time 1 and the means of the attitude scores of their students at time 2 ($r_{12}$) is computed, as well as the correlation between teachers' attitudes at time 2 and the means of the attitude scores of their students at time 1 ($r_{21}$). Then if $r_{12}$ is significantly more positive than $r_{21}$, this is evidence that teachers' initial attitudes had a greater effect on final (mean) student attitudes than initial (mean) student attitudes had on final teacher attitudes. On the other hand, if $r_{21}$ is significantly more positive than $r_{12}$, this is evidence that initial (mean) student attitudes had a greater effect on final teacher attitudes than initial teacher attitudes did on final (mean) student attitudes. A similar approach can be used to study the effects of teacher attitudes or achievement on student achievement, or vice versa. The data collected by Garner (1963), where teachers' and students' attitudes and achievement were measured before and after some treatment time interval, lend themselves to this sort of analysis.

Other data concerning the relationships between teacher characteristics and student attitudes were reported by Alpert et al. (1963). They found that boys' attitudes toward mathematics are more positive when the teacher is more theoretically-oriented and involved, regardless of the teacher's sex. However, there was an interaction between teacher-pupil gender in terms of the effects on student attitudes of more subjective, interpersonal factors such as psychosocial concern. These interpersonal variables were found to have a greater effect on pupil attitudes when pupil and teacher were of the same sex.

Teacher motivation cues. A more recent investigation of the effects of perceived teacher behavior on level of student achievement also found some important sex differences. White and Aaron (1967) classified 185 high-school junior and senior students as achievers, underachievers, and overachievers by the differences in their percentile ranks on the Scholastic Aptitude Test-Mathematical and an objective mathematics achievement test administered at mid-semester. The students also took the Alpert-Haber Achievement Anxiety Test and an opinionnaire designed to assess students' perceptions of the classroom characteristics of their teachers. This procedure was an extension of the McKeachie technique for obtaining measures of four types of motivating cues used by the teacher in the classroom--cues for achievement, affiliation, orderliness, and test and feedback. The data for the six student groups (male and female underachievers, achievers, and overachievers) were analyzed by multiple discriminant analysis of the four measures of teacher-motivating cues and two student anxiety variables. The results show that, in general, girls were more sensitive than boys to the motive-arousing cues of their teachers, and girls were also significantly higher on debilitating anxiety. Girls in all three achievement level groups perceived a lower number of teacher-achievement cues than boys, and there were no significant differences among the three groups of girls on this variable. White and Aaron suggested that teacher-achievement cues were less effective with girls because the girls may already have been at an optimum level of achievement motivation. Other findings were that high-achieving students seemed to be more perceptive of teacher cues emphasizing grades and success in mathematics, but underachievers perceived their teachers as less highly achievement-motivated. Underachieving girls tended to perceive more affiliative, friendly, warm cues and fewer achievement cues from the teachers. Finally, girls in general tended to be more responsive to controlled, conforming behavior on the part of the teacher and to react more to extrinsic rewards and punishments from the teacher.

## Reasons for Liking or Disliking Arithmetic Among Teachers and Prospective Teachers

Assuming that teacher attitudes can be communicated to students and can affect the attitudes and performance of the latter, it may be of interest to determine what percentage of elementary school teachers like or dislike arithmetic and what their reasons are.

Stright (1960) concluded that a large percentage of elementary teachers really enjoy teaching arithmetic and try to make it interesting. But the teacher's age, education, and experience apparently had little effect on her attitude toward teaching arithmetic. It is a reasonable observation, however, that the attitudes of elementary teachers toward mathematics are typically less positive than those of secondary school mathematics teachers (see Wilson et al., 1968, No. 9).

Following a research program initiated some years ago by Dutton (1951), a number of studies during the past ten years have been concerned with the attitudes of prospective elementary teachers toward arithmetic. In a survey at U.C.L.A., Dutton (1962) found that 38% of 127 elementary education majors had unfavorable attitudes toward arithmetic. More recently, Reys and Delon (1968) reported that only about 60% of the 385 University of Missouri education majors whom they surveyed had favorable attitudes toward arithmetic. In Dutton's study (1962), those who disliked arithmetic gave reasons such as: word problems; boring work; long problems; dull; lack of understanding. Those with favorable attitudes pointed to aspects of arithmetic such as: useful, practical applications; definite, precision of concepts; fun just working with numbers. One shortcoming of Dutton's study (1962) is that he attempted to draw conclusions about changes in attitudes over the years since an earlier survey was conducted by using non-equivalent samples. If one finds that a current sample of prospective teachers fills out an attitude inventory differently from an earlier sample, it could mean that attitudes have changed in the intervening years. An equally likely explanation, however, is that the differences are caused by sampling errors.

In a study quite similar to Dutton's (1962) and suffering from some of the same limitations, Smith (1964) compared the attitudes of 123 prospective teachers in the early 1960's with those reported by Dutton for another group ten years before. Among the reasons that Smith's (1964) subjects gave for disliking arithmetic were: lack of understanding; written problems; poor teaching; failure; lack of teacher enthusiasm; too much long work; afraid of it.

In another survey of prospective elementary school teachers' reasons for liking or disliking arithmetic (White, 1963), the most frequent reasons given for disliking the subject were: working word problems; specific skills such as division, fractions, square roots, and per cents; the manner in which arithmetic was taught in elementary school. Prospective teachers indicating more favorable reactions to arithmetic, who were in the majority, gave the following reasons for liking the subject: its challenge; its practical application; its exactness; appreciation of specific skills; solving problems.

The reasons given in these three studies (Dutton, 1962; Smith, 1964; and White, 1963) for disliking arithmetic are quite similar. Some are stimulus variables--word problems, routine, boring work, inadequate teachers, and some are organismic or response variables--failure to understand and fear. A good estimate is that these represent the reactions of approximately one-third of prospective elementary school teachers, and perhaps of college students in general (Dreger and Aiken, 1957).

## Relationships of Prospective Teachers' Attitudes to Their Training

Several investigations have dealt with the relationship between the attitudes and achievements of prospective teachers in teacher-training courses. Unfortunately, the majority of these investigations have employed experimental designs that were inadequate for answering the questions that the investigators posed. The most popular designs--the one-group, pretest-posttest design and the static, two groups comparison--suffer from somewhat different, but equally telling, failures of control (see Campbell and Stanley, 1963). Therefore, the results of these investigations should be viewed as heuristic but not conclusive.

An example of a pretest-posttest study having no control group is that of Reys and Delon (1968), in which the Dutton Attitude Scale was administered to 386 University of Missouri students before and after they took one of three courses in mathematics education. The researchers found a significant decrease from pre- to posttest in the percentage of students agreeing with the following statements on the attitude scale: "I avoid arithmetic because I am not very good with figures," and "I am afraid of doing word problems." An increase was observed in the percentage of students agreeing with the statements: "Arithmetic is very interesting," and "I like arithmetic because it is practical."

Dutton (1965) used a one-group design to assess changes in both attitudes and achievement resulting from intervening instruction. The subjects were 160 prospective elementary school teachers, who were administered an arithmetic comprehension test and an attitude scale as pretests and posttests. Although mean posttest score was significantly higher than mean pretest score on the arithmetic comprehension test, the rise in mean attitude scale score was insignificant. Dutton noted that 25% of the prospective teachers maintained their unfavorable attitudes toward arithmetic in spite of the instructions.

A similar design was employed by Purcell (1964), who was concerned with the relationships of attitude change to increased understanding of arithmetic concepts and to grades in an elementary arithmetic methods course. Although pretest scores in understanding concepts were positively correlated with attitudes and with grades in the arithmetic methods course, there were also a number of negative findings. Pretest attitude scores were not significantly related to grades in the methods course, change in understanding of concepts was not significantly related to change in attitude or course grade, and change in attitude was not related to course grade. However, there were significant improvements in understanding of concepts and in attitudes toward arithmetic.

In still another study along the same lines, Gee (1966) gave pre- and posttests of basic mathematics understanding and attitudes toward mathematics to 186 prospective elementary school teachers in a required mathematics content course at Brigham Young University. The following results were reported: (a) a significant improvement in attitudes toward mathematics and a gain in basic understanding of mathematics by the students while they were enrolled in the course; (b) a significant correlation between pretest attitude and final grades; (c) non-significant correlations between pretest attitude and change in understanding of mathematics; (d) a non-significant correlation between changes in attitudes and changes in understanding of mathematics.

## Attitudes and Training in Experienced Teachers

In order to assess the relationship of amount of teachers' training and experience to their attitudes and understanding in arithmetic, Brown (1961) compared measures of attitudes and achievement in experienced and inexperienced teachers. His findings were that the experienced teachers had more positive attitudes toward arithmetic and a better understanding of basic arithmetic concepts, but no significant relationship was observed between the number of years of teaching experience and either attitude or understanding.

Todd's (1966) purpose was to evaluate the effects of a course, "Mathematics for Teachers," which was taught in various locations throughout the state of Virginia in 1964, on attitude toward arithmetic and change in understanding of mathematics. He concluded that the course produced significant changes in attitudes toward arithmetic and in arithmetic understanding for the teachers who completed the course.

### A Note on Gain Scores

It may be well to insert a note on gains or change scores at this point. Since simple posttest minus pretest difference scores are correlated with pretest scores, initial level of ability, achievement, or attitude is not controlled when simple gain scores are used. One procedure for eliminating the correlation of gains with initial scores is to compute, as a measure of gains, the residual deviations of individuals' actual posttest scores from their predicted posttest scores. The latter are estimated from the regression equation for predicting posttest scores from pretest scores. In a discussion of this procedure, Thorndike (1963) notes, however, that one need not actually compute such residual gain scores in order to apply the concept. A more direct approach is to first find the correlations of pretest and posttest scores with each other and with whatever variable one desires to correlate with the gain scores. Then the part correlation between posttest scores and the third variable, with pretest scores partialed out of the former, is computed (see Thorndike, 1963; pp. 72-74). A similar technique may be used in comparing residual gains on one variable to residual gains on a second variable.

One difficulty with residual gain scores is the "ceiling effect": examinees with high predicted posttest scores will not be able to surpass their predicted scores as much as those with lower predicted posttest scores. Actually, there is no completely satisfactory way to measure gains or changes, but residual gain scores and the associated methods of part and partial correlation are preferable to simple gain scores.

### Two-Group Designs

The two investigations summarized below used a two-group design, which allows for more control over extraneous variables than the one-group design in determining the effects of particular treatments. However, in the studies to be reviewed, the subjects were not assigned at random to the two groups; attempts were simply made to ascertain that the two groups did not differ on variables extraneous to the purposes of the investigations.

Rice (1964) was interested in determining whether formal instruction in modern mathematics influences teacher attitudes toward modern mathematics and toward mathematics in general. He mailed out questionnaires concerning experiences with modern mathematics and attitude toward mathematics to a large number of elementary school teachers in Oklahoma. Four hundred of the 608 replies were analyzed by analysis of variance, chi square, and other statistical procedures. From the results, Rice concluded that teachers who have had formal instruction in modern mathematics have more favorable attitudes toward modern mathematics and toward mathematics in general than teachers who have had no such training. Among the teachers who reported having had training in modern mathematics, there was a significant difference in the attitudes toward modern mathematics in favor of those who had taught in a modern program. Attitudes toward modern mathematics were also more favorable among those who had more training in modern mathematics and among those with more than four years of college. Finally, attitudes toward modern mathematics were found to be unrelated to age, experience, and sex.

Strictly speaking, Rice's (1964) investigation is a correlational study rather than an experiment. Somewhat more "experimental" in nature is the investigation by Wickes (1967), who wished to determine the effects of two different arrangements of courses concerned with concepts in elementary school mathematics on prospective teachers' attitudes and understandings of mathematics. In one arrangement, the completion of a specially designed mathematics course was prerequisite to enrollment in a course in methods of teaching elementary mathematics. A second arrangement was a single consolidated course in which content and methodology were interrelated. The "control" group consisted of 65 students at Baylor University who had taken the first curriculum arrangement in two preceding years, and the "experimental" group was composed of 104 students who completed the consolidated course. Pre- and posttest scores on an attitude scale and a fundamental mathematics concepts test were available for both groups, and it was verified statistically that the two groups were comparable in their pretest scores on these variables. The results showed that both course arrangements produced statistically significant gains in mathematics attitudes and understanding of fundamental mathematics concepts. The control group showed significantly greater gains in understanding of mathematics concepts, but the two groups did not differ in gains on the attitude scale. Wickes concluded that, all things considered, the two-course sequence was more effective than the consolidated course.

In general, the results of the investigations reviewed above indicate that various types of course work in mathematics can affect the attitudes and achievement of teachers and teacher-trainees. But what has recent research to say about the effects of instructional method, i.e., curriculum, on the attitudes and achievement in mathematics of students in the public schools? This is the topic of the next section.

## The School Curriculum

### Rote Memory vs. Meaningful Teaching

In a discussion of a variety of unpleasant experiences in the earlier grades that cause students to avoid high-school mathematics, Wilson (1961) concluded that a primary cause is "drill beyond the fundamental processes." Bernstein (1964) apparently concurred with Wilson's conclusion when he referred to an almost universal agreement among mathematicians and teachers that rote learning procedures are a major factor in producing negative attitudes toward mathematics. Collier (1959) also maintained that teachers should emphasize computational speed less and place more stress on developing mathematical understanding and logical reasoning ability.

Clark (1961) suggested that reliance on rote memory rather than logical reasoning is a consequence of the assignment of formal arithmetic at too early a grade. In his opinion:

> "Children are often confronted in school with situations which few adults would tolerate. Day in and day out there is repetition of meaningless expressions, terms, and symbols. Eventually many children come to dislike arithmetic. Lack of understanding and skills in arithmetic is associated with personality maladjustment and delinquent behavior, including truancy and incorrigibility" [p. 2].

In a study of fourth-grade pupils in a Georgia school, Lyda and Morse (1963) noted positive changes in attitudes toward arithmetic and significant gains in arithmetic computation and reasoning when a "meaningful method" of teaching was employed. The method emphasized the mathematical aim of arithmetic, stressing the concept of number, understanding of the numeration system, place value, the use of fundamental operations, the rationale of computational forms, and the relationships which make arithmetic a system of thinking.

Another way that has been suggested for making arithmetic more meaningful, or at least more interesting, is televised instruction. Kaprelian (1961) administered a questionnaire to 65 fourth-grade pupils to obtain their reactions to the television program "Patterns in Arithmetic." Over 90% of the pupils approved of the program to some extent, and over 75% said they liked arithmetic better after viewing the new arithmetic television program. Finally, 75% of the pupils stated that their attitudes toward arithmetic had changed because the television program helped them understand the subject.

### Effects of Ability Grouping

Grouping pupils in arithmetic classes according to their abilities has frequently been criticized as leading to poor attitudes, either directly or as a result of parental attitudes toward grouping. In order to study the effects of ability grouping on attitudes, Lerch (1961) compared the change in attitudes toward arithmetic of fourth-grade pupils taught intermittently in ability groups with the changes in attitudes of pupils taught in traditional, non-grouped classes. Differences in scores on the pre- and posttest attitude inventories showed that more than half of the pupils in both groups became more favorable in their attitudes toward arithmetic. The average change in attitude of the ability-grouped classes, however, was not significantly different from that of the non-grouped classes. It was concluded that children's attitudes toward arithmetic are less dependent upon classroom organization than on their teachers' attitudes and the methods which the teachers employ.

In another study of the effects of ability grouping, Davis and Tracy (1963) compared the pre- and posttest scores on the California Arithmetic Test of 393 North Carolina fourth-, fifth-, and sixth-graders. The two types of programs were a Joplin-type plan (ability grouping) and a random plan (nonability grouping). It was ascertained that initially the two groups did not differ significantly in their scores on measures of ability, self-concept, anxiety, and attitudes toward arithmetic, which were administered as pretests. Thus, attitude toward arithmetic was a concomitant variable, rather than a criterion variable, in this study. The results were that pupils in the Joplin-type plan did not gain significantly more in arithmetic achievement than pupils in the random plan. Consistent with the conclusion of Lerch (1961) referred to above, Davis and Tracy (1963) concluded that differences among teachers in their knowledge of arithmetic, attitudes toward arithmetic, and their variability in method of teaching--factors which were not controlled or measured in this study--are important variables to consider in future research on ability grouping.

## School Mathematics Study Group (SMSG) Curriculum

In a discussion of motivations in mathematics, Bernstein (1964) suggested that organization of subject matter, such as that in SMSG, may improve attitudes toward mathematics. Unfortunately, studies have typically failed to verify Bernstein's suggestion; the teacher, rather than the curriculum, still appears to be the more significant variable.

For example, in a comparison of SMSG and non-SMSG seventh-grade classes, Alpert et al. (1963) observed that the SMSG curriculum did not increase students' positive feelings toward mathematics, either absolutely or when compared with the non-SMSG curriculum. However, teachers with a highly theoretical orientation tended to produce more positive feelings in SMSG classes, but not in non-SMSG classes. Concerning measures of attitudes, Alpert et al. (1963) found that the attitudes toward mathematics of SMSG students became less positive from fall to spring testing, whereas the attitudes of non-SMSG students remained relatively constant.

Similar results have been obtained in other investigations which have compared SMSG and traditional curricula in elementary and junior high school (Hungerman, 1967; Osborn, 1965; Phelps, 1963; Woodall, 1966). In general, these studies have found that the mean mathematics attitude scores of students taught by the SMSG curriculum is not significantly different from, or even more negative than (Osborn, 1965), the mean attitude score of students taught mathematics by the traditional curriculum. With respect to achievement, one study found results favoring the SMSG curriculum (Osborn, 1965), the results of another favor the traditional curriculum (Hungerman, 1967), and still another found no significant difference between the two types of program (Woodall, 1966). In general, scores on conventional standardized tests of achievement in mathematics tend to favor the traditional, non-SMSG curriculum, whereas scores on special tests like those constructed by the School Mathematics Study Group for use with its materials tend to favor the SMSG curriculum.

But why does the SMSG curriculum fail to produce more positive attitudes toward mathematics, as Bernstein (1964) hoped that it would? Osborn (1965) suggested that this can be explained by the fact that the SMSG curriculum is more abstract and demanding than the traditional curriculum, causing the students' attitudes to fail to change at all or to even become more negative as the length of time that they study the SMSG program increases.

Before one goes too far in interpreting the above results, however, it should be emphasized that in these investigations the available subjects were not assigned at random to the two types of curricula. The investigators merely analyzed data obtained from existing groups, in some cases attempting to assure themselves that the groups did not differ significantly in their pretest scores, in other cases using analysis of covariance as an attempt to control for initial group differences. But without random assignment of subjects to conditions, there is little control over extraneous variables, and without such randomization analysis of covariance is not a legitimate statistical procedure. Therefore, many of the conclusions of the studies reviewed above must be viewed as tentative until more controlled research is done.

### Other Modern Mathematics Programs

Correlational data which show that students in a special public school program have more positive attitudes toward the subject than students in other types of programs are plentiful. But of course it is quite possible that the students in the special program were attracted to or selected for the program to begin with because of their positive attitude toward the subject. A case in point is the finding of Ellingson (1962) that high school students in college preparatory classes had somewhat more positive attitudes toward mathematics than students in terminal or general math classes. The self-selection factor, where those with more positive attitudes and higher ability elect the special course, and the effects on morale of being in a "new" program (the Hawthorne effect) undoubtedly influence the results of investigations in which there is no true control group.

Research designs similar to those of studies reviewed in the previous section on the SMSG curriculum have been employed to compare other mathematics programs with the traditional program. For example, Comley (1966) compared the college mathematics achievement and attitudes of students who had the University of Illinois Committee on School Mathematics (UICSM) program in public school with those of students who had traditional high school mathematics. Students in a number of colleges were administered a mathematics attitude questionnaire, and other data concerning total number of terms of college mathematics taken, major field of study, number of elective mathematics courses taken, types of mathematics courses taken, and overall mathematics averages were obtained from transcripts and questionnaires. After the criterion scores of the UICSM and non-UICSM groups were adjusted by covariance analysis on numerical and verbal aptitudes, high school grade averages, school size, and percentage of each school's students who went to college, there were few differences between the two groups in college mathematics achievement. The UICSM group did take significantly more college mathematics however, and did as well as the non-UICSM students. In addition, the UICSM students had significantly more favorable mathematics attitudes than the non-UICSM group.

In an investigation by Yasui (1967), a modern-mathematics group studied the Secondary School Mathematics textbook series in grades 10, 11, and 12 in the Edmonton, Alberta public schools. A control (traditional) group consisted of 125 students selected from high schools not exposed to modern mathematics. After "adjusting" for individual differences in scholastic ability with ninth-grade scores on the School and College Ability Test, the mean score of the modern-mathematics group was significantly higher than that of the traditional group on test items of the Contemporary Mathematics Test which contained material common to both curricula. Although the difference between the mean scores of the two groups on an inventory of attitudes toward mathematics was not significant, attitude scores were significantly correlated with achievement in both groups. It may be observed that the two groups in this study were not equated for initial attitude toward mathematics, so it is not certain what the failure to find a significant difference in mean attitude in the twelfth grade means. Perhaps the two groups had equivalent mean attitude scores to begin with and both became more positive, more negative, or remained the same. Or perhaps one group had more positive attitudes than the other at the outset, and the initially more positive group became more negative, or the initially more negative group became more positive.

The aim of a recently completed project by Ryan (1967) which involved 126 pairs of mathematics classes in schools distributed over a five-state area, was to compare the effects of three experimental "modern" programs in secondary mathematics--the Ball State, UICSM, and SMSG programs--on the attitudes and interests developed in ninth-grade pupils. Self-report measures of attitudes and interests were administered to the students at the beginning and end of the school year, and systematic observations of behavioral signs of student interest were also made. Pupil characteristics such as sex and achievement level, and teacher characteristics such as experience with the programs were considered in the data analysis. The general finding was that the experimental programs, when compared with conventional mathematics programs, had little differential effect on the attitudes and interests of the pupils. There was a slight tendency, however, for the Ball State program to be related to the development of less positive attitudes and the UICSM program with more positive attitudes toward mathematics, when compared to conventional programs. The less positive attitude of the students using the Ball State program was associated with the reported greater difficulty which students had in understanding these materials. Measured pupil and teacher characteristics did not interact significantly with type of program in determining its influence, but change in attitude was generally related to change in grade received relative to the previous year and to the degree of difficulty which pupils experienced with the materials.

## Other Curriculum Comparisons

Especially noteworthy for its attempt to control extraneous variables is an investigation by Devine (1967), who compared program-centered with teacher-centered teaching of first-year algebra. There was a random selection of two classes (but not a random selection of subjects for the classes)--an experimental and a control class--in each of two high schools. Achievement and attitude tests were administered at various times during the school year. A result of the study which is particularly interesting is the obtained interaction between teacher experience and type of curriculum in their effects on student achievement in mathematics. When the teacher was experienced, the mathematics achievement of the program-centered group was lower than that of the teacher-centered group; there was no change in either group, however, in attitudes toward mathematics or toward programmed materials. When the teacher was inexperienced, the program-centered group achieved as well as the teacher-centered group, but the attitudes toward mathematics and toward programmed instruction became more negative in both groups. In a summary of the results, Devine (1967) concluded that when an average or above average teacher is available, greater achievement is obtained in a conventional, teacher-centered classroom approach.

A final investigation concerned with curriculum is that of Maertens (1968), although the experiment is cited as an illustration of a controlled design as much as for the specific results. The experiment was designed to assess the differential effects of the curriculum practice of assigning homework in arithmetic on the attitudes of third-grade pupils toward school, teacher, arithmetic, homework, spelling, and reading. There were three treatments--control (no homework), common practice (regular teacher assigns homework), and experimenter-prepared homework. Pupils were randomly assigned to three classrooms within each of four schools, and within each classroom pupils were assigned to three intellectual ability groups. The data from five subjects in each of the three ability groups within each of the 12 classrooms were analyzed by analysis of variance of a Latin square design with repeated measures. The basic data were scores on the six measures of attitudes administered at the end of each of three 3-month treatment periods. Since there were no statistically significant differences among the three treatments, Maertens concluded that arithmetic homework does not uniformly affect pupils' attitudes toward arithmetic and the other five sources referred to above. Consequently, teachers need not omit purposeful arithmetic homework as a general practice because of fear that it may create negative pupil attitudes.

## Developing Positive Attitudes and Modifying Negative Attitudes

Alpert et al. (1963) made a number of suggestions, growing out of their research on the SMSG program, for further improvement of achievement in and attitudes toward mathematics, viz.: (a) more attention by textbook writers to those aspects of school which affect psychological determiners of success in mathematics; (b) more attention to teacher selection and training and to the possibility of taking into account teacher characteristics when grouping pupils; (c) consideration given in course design to the meaning of education in mathematics for women; (d) communication to parents about the nature of the effects which they have on children's mathematics education. These are commendable goals on a broad scale. As noted by Bassham, Murphy, and Murphy (1964), in order to change a pupil's attitude toward mathematics his perception of himself in relation to mathematics materials must be changed. Therefore, what have other mathematics educators and researchers recommended and accomplished in the effort to change students' perceptions of themselves in relation to mathematics?

## Emphasis on Relevance, Meaningfulness, and Games

Since the time of John Dewey, there has been a growing emphasis on the need to make education practical or relevant. Nevertheless, Bernstein (1964) argued that educators have failed to stress sufficiently the use of mathematics for studying and controlling our physical and social environment. In one of the reports of the Ohio State University Development Fund, Nathan Lazar maintained that children can learn to love arithmetic if they are not bombarded with meaningless memory drills. His approach to helping children understand arithmetic is to use simple reasoning problems about black horses, brown cows, and white sheep. In addition, he invented a game-like apparatus called an "aba-counter"--a variation on an abacus with multi-colored beads strung on rods--to help make mathematics more meaningful. Tulock (1957) also recommended that games, contests, and audio-visual aids be used to heighten interest in mathematics.

Zschocher (1965) experimentally investigated the effectiveness of various group mathematical games on the performance of first-grade children in day care centers in Germany. For a period of five months, 70 girls and boys were given the opportunity of playing the games before and after classes. The results were that their scores on standard tests of number concepts, spatial orientation, and basic arithmetic rose significantly. A control group of 75 subjects showed no significant improvement on the tests. However, teachers did not discriminate significantly between the experimental and control children in their evaluations of the children's mathematical achievement. In a related study on older children, Jones (1968) obtained a significant improvement in the attitudes of ninth-grade students in remedial classes when they were taught mathematics by modified programmed lectures and mathematical games.

## Providing for Success Experiences

Many writers (e.g., Lerch, 1961; Tulock, 1957) have observed that pupils who constantly fail mathematics lose self-confidence and develop feelings of dislike and hostility toward the subject. In order to cope with such negative attitudes, the teacher must provide for success experiences in the learner; the child should be taught to set reasonable goals that culminate in the reward of success. The need to provide for success experiences was also referred to by Proctor (1965) in a discussion of techniques for giving self-confidence and faith to slow learners and thus changing their attitudes toward mathematics.

## An Experiment on Mediated Transfer

It is not particularly surprising that success in mathematics, which is a pleasant experience, can cause a person's attitude toward the subject to become more positive. But although it is not always possible for an individual to succeed in mathematics, the results of an experiment by Natkin (1966) suggest that simply getting him to associate mathematics with something pleasant may alter his attitude or anxiety toward the subject.

The initial group of subjects selected by Natkin were male and female undergraduates who had scored above the mean on the verbal section and one standard deviation below the mean on the mathematical section of the Scholastic Aptitude Test. The first step in the experiment was to determine the galvanic skin responses (GSR's) of the subjects to mathematics and non-mathematics stimuli. Then those subjects whose GSR's to the mathematics stimuli were significantly greater than their GSR's to the non-mathematics stimuli were randomly assigned to experimental and control groups. In the first stage of the experiment proper, the subjects in the experimental group learned, by a paired associates procedure, to associate the mathematics stimuli with nonsense syllables; in the second stage they learned to associate the same nonsense syllables with strongly pleasant phrases. The subjects in the control group learned the same mathematics stimuli-nonsense syllable pairs in the first stage as the experimental group, but the former learned nonsense syllable-neutral stimuli associations in the second stage. As was predicted, scores on a test of anxiety toward mathematics showed a more significant decrease from pre- to post-experimental testing in the experimental group than in the control group. The post-experimental test of anxiety was administered only five minutes after the learning session, however, and one might well question the permanence of the decrease. Other questions which need to be answered are whether the anxiety change observed by Natkin (1966) would have generalized to other situations (e.g., school tests) involving mathematics and whether his "mediated transfer" procedure can also affect performance in mathematics. In any event, Natkin concluded that the experimental procedure created a mediated "therapy" effect on mathematics anxiety, quite similar to the desensitization of fears by behavior therapy. He also noted from the response patterns in the test data that early traumatic learning was largely responsible for anxiety toward mathematics.

Natkin's (1966) experiment is important because it showed, by means of a well-controlled experiment, that it is possible to affect anxiety toward mathematics, if only for a short time. There are behavior therapy techniques other than "mediated transfer" that should certainly be explored, as methods for reducing mathematics anxiety. But anxiety is a purely emotional reaction, and although attitude--the topic of the present review--is partly emotional, it also has a cognitive component. Therefore, techniques for attacking the cognitive aspect of negative attitudes, as well as their emotional aspect, are needed. This involves more research and, consequently, brings us to the closing section on evaluation of previous research and suggestions for future research on attitudes toward mathematics.

## Critique of Previous Studies and Suggestions for Further Research

### Criticisms

A number of critical comments about previous research concerned with the determiners and effects of attitudes toward mathematics have already been made in this review. Some of these criticisms apply as well to other areas of educational research, and they have been widely recognized. In general, there has been too much reliance on correlational methods and on indirect measures of behavior, such as questionnaires and other student-reports. It is admittedly easier to point to a need than to satisfy one, but the superabundance of correlational results which have been reported is primarily heuristic; controlled experiments are needed to test the hypotheses suggested by the significant correlation coefficients. Analysis of covariance is of no help unless investigators randomly assign subjects to groups. The procedures of "matching" and "statistical control of concomitant variables" should not be viewed as substitutes for random assignment of subjects to treatment conditions. As was noted previously, random assignment is an even more important assumption of analysis of covariance than it is of analysis of variance.

In their treatise on research methods in education, Campbell and Stanley (1963) have discussed these matters at length, detailing the sources of error left uncontrolled in various research designs. The proposal of Campbell and Stanley (1963) for obtaining information concerning cause-effect relations through correlations across time (cross-lagged panel correlations) would appear to be a potentially fruitful approach to an analysis of the direction of cause and effect in teacher-pupil attitudes and achievement.

Whenever correlational methods are to be used, and especially correlations among gain scores, the investigator should first become familiar with Thorndike's (1963) discussion of methodological problems in research on over- and under-achievement. Of particular importance when designing research to determine whether attitude variables are related to achievement or other variables in any significant way are the sections of the Thorndike book which are concerned with part correlation and gain scores.

The remainder of the review will deal with some suggestions for further research on attitudes toward mathematics--research which it is hoped will take into account both the findings and shortcomings of the work that has been reviewed in preceding sections of this paper.

Measures of Attitudes

Since the usefulness of the results of research is frequently limited by the precision with which outcomes are measured, something needs to be done to improve the accuracy of measures of attitudes. The task may be approached in several ways. Anttonen (1967), for example, has pointed to the need for research aimed toward improving the readability of attitude measurements at the elementary school level. In addition, the reviewer feels that the concept of a general attitude toward mathematics should be supplemented with that of attitudes toward more specific aspects of mathematics, for example, problem-solving and routine drill. This is similar to a recommendation made by Moss and Kagan (1961) with respect to the concept of achievement.

One possible approach in designing such multivariate attitude instruments is to follow a stimulus-response model like that proposed by the reviewer (1962) and by Endler, Hunt, and Rosenstein (1962) for the concept of anxiety. Such instruments should be of greater diagnostic usefulness than the current scales of general attitudes toward mathematics with their single, overall score. The stimulus-response approach could also consider the distinction between the cognitive and emotional components of attitudes in the design of attitude instruments.

Teachers

Although it is certainly unfair to indict teachers too strongly as creators of negative student attitudes toward mathematics, the results of research have suggested that the teacher, perhaps even more than the parents, is an important determiner of student attitudes. As noted by Banks (1964, pp. 16-17):

> An unhealthy attitude toward arithmetic may result from a number of causes. Parental attitude may be responsible.... Repeated failure is almost certain to produce a bad emotional reaction to the study of arithmetic. Attitudes of his peers will have their effects upon the child's attitude. But by far the most significant contributing factor is the attitude of the teacher. The teacher who feels insecure, who dreads and dislikes the subject, for whom arithmetic is largely rote manipulation, devoid of understanding, cannot avoid transmitting her feelings to the children... On the other hand, the teacher who has confidence, understanding, interest, and enthusiasm for arithmetic has gone a long way toward insuring success.

In order to provide more information on the effects of teacher attitudes, more direct measures of teacher attitudes and their consequences, for example by classroom observation, should be obtained. Student reports of perceived teacher attitudes and also teacher reports of their own attitudes are useful, but direct observation of teacher-pupil interaction in mathematics classes is also needed. In addition, more attention should be given to the mathematics training of elementary-school teachers. If the law of primacy holds, the influence of elementary teachers on pupil attitudes should be even greater than that of secondary teachers.

Finally, it would be interesting to conduct a Rosenthal-type investigation to determine the effect of teacher expectations in mathematics on student attitudes and achievement. Using the procedure of Rosenthal and Jacobson (1968), after the students are tested initially the teachers would be informed that one group of children--actually selected at random--will show an increase in mathematics achievement and/or positive attitudes toward mathematics during the following semester. Measures of change in student achievement and attitudes in both the experimental and control groups would be assessed to determine the effects on these variables of teachers' expectations.

## Longitudinal, Multivariate, and Experimental Studies

Alpert et al. (1963) and Anttonen (1967) have pointed to the need for longitudinal reasearch on patterns of performance in mathematics emerging over time, and on psychological variables related to these changes. Anttonen (1967) maintained that the measurements of attitudes and achievement in such studies should be over a period shorter than the six-year span which he used. A period of one or two years, as in the NLSMA studies, would probably be most satisfactory. However, there is a need for both longitudinal and "one-shot" studies to determine the effects and interactions of many variables--teachers, parents, curriculum, and such pupil variables as general and special abilities, biographical factors, interests, and personality characteristics--on attitudes and performance. The implication of much of what has been said previously in this review is that multivariate programs should not be limited to correlational designs. Weaver and Gibb (1964) have called for research on the genetic development of mathematical ideas and abilities among children in different instructional conditions and in different mathematical environments. They maintain that since the personality characteristics of children, instructional methods and materials, school organization, motivating conditions, and level and sequence of mathematical content interact to such a degree, multivariate studies rather than studies of the effects of only one of these variables are necessary. This type of program calls for a marriage of the correlational and experimental approaches to research.

Alpert et al. (1963) have cited further classroom experimentation on the use of self-instruction programs, modern mathematics programs, specially trained teachers, use of innovative teaching techniques, and training films as important needs in research and development in mathematics education.

### Development and Modification of Attitudes

As this review has shown, there has been only a small amount of research on techniques for developing positive attitudes and modifying negative attitudes toward mathematics. Bassham, Murphy, and Murphy (1964) point to the desirability of further experimental work to explore the development and/or modification of anxiety, attitudes, and other variables which affect achievement in mathematics.

It is clear that serious thought must be given to experiments concerned with the temporary and more permanent effects of preschool and early school experiences on attitudes toward and performance in mathematics. In both the development and modification of attitudes, and in training and remedial work, a question is how to make mathematics more interesting. New methods may be initially motivating ("Hawthorne effect"), but their effects will not last if the teachers are poorly trained, the parents are not sympathetic, and the students are not sucessful in mastering the subject.

## Summary

Over three dozen journal articles, two dozen doctoral dissertations, and a half-dozen reports of studies concerned with attitudes toward mathematics which have been written during the past decade were reviewed. The major topics covered were: methods of measuring attitudes toward arithmetic and mathematics; the grade distribution and stability of mathematics attitudes; the relationships of attitudes to achievement in mathematics; the relationships of mathematics attitudes to ability and personality factors, to parental attitudes and expectations, to peer attitudes, and to teacher characteristics, attitudes, and behavior. Also discussed were investigations dealing with the effects of modern mathematics curricula and other curriculum practices on attitudes. Of all the factors affecting student attitudes toward mathematics, teacher attitudes are viewed as being of particular importance. Finally, research on techniques for developing positive attitudes and modifying negative attitudes was summarized.

Among the criticisms made of research on attitudes toward mathematics were the use of crude measures of attitudes, excessive reliance on correlational methods, improper use of covariance analysis, inadequate control of extraneous variables, and failure to use adequate measures of gains. Suggestions for further research included adequate familiarization with previous studies concerned with the topic, the development of multifaceted measures of attitude, more extensive multivariate experiments extending over longer periods of time, and more attention to techniques for developing positive attitudes and modifying negative attitudes.

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# CLASSROOM TEACHING OF MATHEMATICS[1]

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What *is* a good mathematics teacher? What are the personal characteristics of a good mathematics teacher? What teaching method is most effective in mathematics classes? What sort of preservice and inservice education is most appropriate for mathematics teachers? Few questions in mathematics education are more important and few generate more vigorous debate or consistent disagreement. But despite long and active interest in the problems, research offers few important guidelines in the search for personal attributes, classroom styles, or educational preparation of successful teachers. Extensive investigation has failed to show significant or consistent correlation between fundamental characteristics of teachers--such as experience, knowledge of mathematics, collegiate preparation, or attitudes toward mathematics--and the achievement or attitudes of their students. Evidence from comparisons of two or more teaching methods supports no one method as superior in mathematics teaching.

Much current research in mathematics teaching continues the traditional approach to finding the elusive "good teacher." There is growing evidence, however, of creative, yet careful, new research strategies and techniques.

## *Weaknesses and Strengths of Current Research*

Attempts to identify critical teacher personal characteristics of teachers have been limited by narrow vision of what the key variables might be and by oversimplified measures of effectiveness. Several recent studies have uncovered previously unexamined teacher attribute dimensions that appear to have promise in predicting effectiveness. Others have attempted to broaden the accepted criteria of effectiveness; instead of a simple *good* or *bad* appraisal of effectiveness based on standardized student achievement tests, these investigations have sought a detailed description of student outcomes--cognitive and affective, manifest and latent.

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[1] A shortened form of this paper, entitled "Research in Classroom Teaching of Mathematics," will appear in the *Review of Educational Research*, October 1969.

Inadequacy of criterion measures has also plagued research in methods of teaching mathematics. Instead of attempting a careful description of the outcomes associated with particular methods of instruction, methodology research has usually been an attempt to show the superiority of one name method over another, as measured by achievement tests. This approach is insensitive to the differential effects that each individual treatment might have. As a consequence, results from methods of instruction research are inconclusive; the strong and weak features of each method tend to counterbalance each other. Current research in mathematics teaching methods is focusing on the crucial problem of aptitude-treatment interaction (ATI) in an attempt to discern the unique impact of various teaching styles.

Methodology research has also suffered from the fact that most studies are small scale projects, often a single investigator teaching one or two experimental classes and a like number of control classes while assuming responsibility for evaluation of the results. Thus even when significant results are reported, the impact of such findings on teaching practice is minimal; the investigator's personal involvement in all aspects of the research makes bias inevitable, and the methods being compared are seldom described in behavioral terms precise enough to allow repetition of the method in other classes with other teachers. Current research, with the benefit of sophisticated technology for recording classroom activity, is making a strong bid to sharpen definitions of teaching methods and thus make methodology research more reliable.

New approaches to the study of mathematics teaching are fresh steps into what has been an extremely discouraging area of research. An equally important development is the growing interest in a theory of mathematics instruction. Even if the various investigations relating to mathematics teaching are successful in describing relationships among pairs of key instructional variables, one more important task remains--carrying these findings into widespread classroom practice. Implementation of teaching research requires integration of all the relevant knowledge into a coherent prescription of instructional behavior. Tentative theories of mathematics instruction are beginning to provide a conceptual framework indicating important research questions and the relationships of widely varied research findings to classroom teaching.

In the past, this translation of scattered research results into teaching practice has been left largely to the ingenuity of individual teachers in the classroom. Thus Henderson (1963, p. 1025) concluded, with wide support, that as of 1963 (and probably at the present time, also) mathematics teaching was an art, not a science, and that

> for it to move toward becoming a science, we need much more empirical research to test current theories. But we also need new theoretical concepts or orientations that will provoke different questions to be asked.

Research in mathematics teaching since that date shows definite signs of meeting his demands for rigorous empiricism and new theoretical organization.

Much of the credit for new theoretical and empirical vitality in research is due to the appearance in 1963 of Gage's Handbook of Research in Teaching. The influence of research guidelines in the Handbook is not noticeable in early research covered by this review, but recent studies are increasingly well designed and administered.

This review covers research in mathematics teaching reported from 1964 through 1968. Since comprehensive listings of research have become annual features of several journals, the discussion here will not be a complete catalog of research. Instead, it will indicate significant results and promising directions for study. The review is divided into three major areas: (1) personal characteristics of teachers; (2) teacher classroom behavior, including methods of teaching; and (3) teacher education.

## Characteristics of Effective Teachers

Although previous investigations of teacher characteristics and effectiveness had failed to yield a consistent or definitive profile of the effective mathematics teacher, introduction of strikingly new mathematics curricula provoked renewed interest in teacher characteristics research during the past ten years. The new curricula brought new content, greater mathematical precision, and a concept (rather than skill) orientation to mathematics study and teaching. These changes reopened several fundamental questions:

1) Is there an identifiable relationship between teacher knowledge of mathematics and student achievement?
2) Are teacher age or years of experience factors which influence student achievement?
3) How do teacher attitudes toward the new mathematics affect student attitudes or achievement?

## Minnesota National Laboratory Study

The most comprehensive, sophisticated, and imaginative investigation of teacher characteristics and student achievement was conducted by the Minnesota National Laboratory for the Improvement of Secondary Mathematics as part of its effort to evaluate experimental curricular materials of the School Mathematics Study Group. SMSG was interested in determining whether the new curricular materials it was producing were suitable for widespread implementation by teachers of varying ability and qualifications.

An exploratory study involving 21 teachers of Grades 6-8 during the 1958-59 school year and another involving 127 teachers of Grades 7-12 during the 1959-60 school year failed to find any significant correlation between students' achievement and the experience, collegiate courses or grades, and professional activity of their teachers. However, analysis of daily logs kept by the participating teachers during the 1958-59 study and of monthly checklist reports made during 1959-60 indicated that the most effective and least effective teachers could be differentiated by measures of productive thinking--the ability to generate ideas about success or failure of various teaching lessons and ideas about alternative procedures for teaching particular concepts.

These preliminary investigations led to design of the main study (Torrance and Parent, 1966) which sought further evidence on the correlates of teaching effectiveness that might be described as qualifications to teach--experience, collegiate preparation, and professional activity--as well as the importance of variables in teacher-pupil interaction and teacher productive thinking ability. During the 1960-61 school year and again in 1961-62, each of 63 teachers taught the experimental SMSG course to one class.

Effectiveness of this teaching was evaluated from the following aptitude and achievement testing program:

Fall: _School and College Ability Tests_ (Grades 7 and 8) or _Differential Aptitude Tests_ (Grades 9-12) and _Sequential Tests of Educational Progress in Mathematics_, Form A.

Spring: _Sequential Tests of Educational Progress in Mathematics_, Form B.

Measures of the variables possibly correlated with achievement were obtained from both students and teachers:

Students: Responded to several student attitude inventories and a student checklist of learning activities.

Teachers: Completed questionnaires giving information about experience, education, and professional activity and also completed nine end-of-the-month reporting forms designed as tests of productive thinking. A teacher and pupil activity checklist gave information about procedural variables.

The teachers were divided into three groups according to the measures of effectiveness and a wide variety of correlations were examined to identify factors differentiating the most and least effective groups.

Results from both the 1960-61 and 1961-62 trials confirmed the earlier conclusion that effectiveness of teachers using the SMSG materials is not significantly correlated with teachers' experience, collgiate courses and grades, or participation in professional activities. Most and least effective teachers were not differentiated by the amount of time they spent in preparation for teaching. There was only a weak indication that procedures in making assignments, explaining new material, conducting learning and thinking experiences relevant to previously assigned material, and evaluating and responding to student performance make a difference in teacher effectiveness. Application of Flanders' system of Interaction Analysis failed to locate significant differences in the patterns of classroom behavior developed by effective and ineffective teachers.

The most significant predictor of teaching effectiveness was the productive thinking ability of teachers. The most effective teachers produced more ideas about indications of success or failure in their teaching, causes of success or failure, and alternative ways of teaching course concepts. The student attitude and learning activities questionnaires revealed a number of significant indicators of teaching effectiveness. This suggests that teacher effectiveness is intimately connected with pupil attitudes and perceptions concerning the methods of their teachers, the school, text materials, and the class as a group.

Thus the Minnesota study confirms earlier indications that the search for predictors of effective teaching must move beyond the gross measures of ability and formally identifiable qualification. Apparently, qualification beyond certain minimal standards is not reflected in greater effectiveness. In the search for more subtle yet penetrating factors in teaching success, the study provides fresh insight into topics and techniques for future research.

## Secondary Mathematics Teachers

One of the teacher variables measured only indirectly in the Minnesota study is the teacher's knowledge of contemporary mathematics--a factor that would logically seem correlated with effectiveness. In a doctoral study, Massie (1967) asked a panel of 12 mathematics educators to identify the characteristics of contemporary mathematics. He then developed and pilot tested an examination called Contemporary Mathematics: A Test for Teachers. Drawing a sample of 273 prospective teachers in college in eight states and 58 teachers with experience in modern mathematics, he showed that scores on the contemporary mathematics test correlated significantly with the quantitative, verbal, and total scores on the Henmon-Nelson Test of Mental Ability, and he gathered some descriptive data about the mathematical competence of the chosen sample.

In another doctoral study Lyng (1968) used Massie's test to investigate the relations between teacher knowledge and (1) semseter hours of academic preparation, (2) years of experience teaching contemporary mathematics, (3) semester hours of student teaching, (4) semester hours in methods courses, (5) years since bachelors degree, (6) age of teacher, (7) grade point average for recent course work, and (8) ability to read and comprehend unfamiliar mathematics. He found the best predictors of teacher knowledge of contemporary mathematics were variables 1, 6 (negative correlation), 2, and 8. The multiple correlation coefficient for these variables was .74 ($p < .01$). But even if these results are shown to indicate causal relations between teacher attributes and knowledge of contemporary mathematics, they leave open the question of whether teacher knowledge is critically related to teacher effectiveness.

Kennedy (1963) probed another kind of teacher characteristic that should have a bearing on effectiveness. He developed a test of skill in solving mathematics teaching problems. The test consisted of 17 tape-recorded problem situations, excerpted from actual algebra classes, each of which was followed by oral questions about solutions. A panel of experts devised rating scales for responses, and the test was administered to 311 teachers of varying backgrounds.

Scores on the test differentiated groups of pre- and inservice teachers at the .01 level of confidence in the following order from poorest to best: (1) non-mathematics majors, (2) elementary education majors, (3) preservice mathematics teachers without methods course experience, (4) preservice mathematics teachers who had had a methods course, (5) preservice mathematics teachers who had practice teaching experience and a methods course, and (6) experienced mathematics teachers. There was no way, however, to consider the relationship between differences on the test and teaching effectiveness.

## Elementary Mathematics Teachers

The initial thrust of recent curriculum innovation was in the secondary school, but emphasis soon shifted to the elementary school. Elementary school teachers, already faced with the task of specializing in every subject of the curriculum, have been asked to learn a great deal of new mathematics and to learn and teach the traditional content from a new point of view. Concern for teacher's subject-matter competence has made teacher ability a popular area for research.

Several studies have attempted to assess the mathematical competence of pre- and inservice elementary school teachers. The results consistently show that teachers do not have the knowledge of modern mathematics expected as a prerequisite of effective teaching. Melson (1965) administered a 33 item test of modern mathematics to 41 beginning teachers in the Philadelphia area. She found a median score of 12 correct and 27 teachers scoring below 50 percent correct. The test does seem to emphasize the language rather than the substance of contemporary mathematics, and the results might thus reflect slow change in the college preparation of elementary teachers. F. Smith (1967) showed that an appropriately designed content course would produce dramatic changes in scores on the Melson test.

In a penetrating study of prospective teachers in Negro private colleges, Carroll (1964) compared the mathematical knowledge of 358 teachers just before graduation from college with normative data (STEP tests) and analyzed the particular strengths and weaknesses in the knowledge of these teachers. He found that the groups functional competence in mathematics was at the level of seventh and eighth grade students, with knowledge of geometry and probability the weakest. In another more limited study, Williams (1966) found teacher preparation in arithmetic generally disappointing.

The grim picture of mathematical competence among elementary teachers, as reported during the 1964-68 period, continues a long standing pattern. But it also indicates slow awakening by colleges to the needs of these teachers faced with a curriculum vastly different from the one they experienced in their own schooling or college preparation. In the elementary school, as in the high school, the question of relations between mathematical competence and student achievement remains unanswered. The available information is not encouraging.

R. Moore (1965) examined the hypothesis that there is a positive relation between level of teacher understanding (as measured on the Glennon Test of Mathematical Understanding) and student achievement (as measured by the SRA Arithmetic Series). Using 10 fourth grade and 11 sixth grade classes as subjects, he found no significant correlation between mean classroom gain in achievement and teacher understanding in either fourth or sixth grade classes. He did discover a significant correlation between the within class variability of achievement and teacher understanding, with more knowledgeable teachers producing greater variability. This correlation was not significant at the sixth grade, but there, limitations imposed by the ceiling of the achievement test were a factor preventing significance.

In another doctoral study, Lampela (1966) found little evidence of a correlation between teacher understanding and change in pupil understanding of certain concepts in elementary school mathematics. W. Smith (1964) looked at interrelationships among arithmetic achievement of eighth graders and patterns of teacher preparation. He found special subject matter preparation and years of teaching experience were not independent predictors of student achievement, but professional education preparation of teachers apparently was such a predictor.

Godgart (1964) examined the connection between arithmetic problem-solving abilities of teachers and the growth in problem-solving ability of their students. He found no significant relation.

Yet another discouraging report comes from Rouse (1967) who examined correlations between academic preparation of elementary teachers and achievement of their students. Teacher attributes were high school and college preparation; student achievement was measured in arithmetic reasoning and fundamentals, as determined by examination of permanent records. In three groups--Grades K-4, K-6, and K-8--Rouse found almost no correlation between teacher attributes and student achievement.

As is the case with teacher characteristics research at the secondary level, it appears that predictors of teaching success will not be found among the obvious variables of experience, academic preparation, and mathematical knowledge. It might be that the mathematical knowledge of elementary teachers is uniformly so low that it does not play an important role in effectiveness.

This should be watched as the academic preparation of teachers improves, under influence of new teacher education proposals.

A more sophisticated approach to the identification of effective elementary school teachers is illustrated in a study by Peskin (1964). On the basis of standardized tests, she classified attitudes and mathematics understanding of a group of seventh grade teachers as high, middle, or low. She then made a similar appraisal of student achievement and attitudes and investigated the interactions among teacher and student attributes. This analysis produced several interesting and significant correlations. For example, teachers with middle range attitudes toward mathematics and high mathematical understanding seemed to produce best student achievement in geometry. Teachers with low attitudes and high understanding produced poorest student achievement in arithmetic. This type of knowledge-attitude-achievement interaction looks like a promising source of deeper insight into the teacher characteristic problem.

Turner (1964) reported progress in a different kind of search among complex interactions of teacher variables. He analyzed teaching as consisting of three types of work tasks: (1) the setting of tasks for pupils, (2) appraisal of pupil responses according to criterion responses, and (3) instructional tasks designed to close the gap between observed and expected student performance. Next he devised a paper and pencil test of teacher problem-solving ability, *Mathematics Teaching Tasks*, (MTT), which was shown to differentiate between experienced and inexperienced teachers.

In the main study, involving beginning teachers in 12 Indiana school districts, Turner sought significant interactions among scores on the *MTT*, teaching success (measured by supervisor ratings), and various socioeconomic indices (measured by Ryan's *Teacher Characteristic Schedule*) concerning the teacher and the institutional context within which he worked. Although the study was not considered conclusive in any of its findings, it suggested a variety of hypotheses about the way teacher success and *MTT* ability change during first years of teaching and the ways these factors are influenced by the socioeconomic makeup of the community in which the teacher works.

In particular, the combination of teacher characteristics that will be associated with success in a given institutional context appears to depend primarily on the socioeconomic composition of the student population. When a large proportion of students are of working class background, performance of the teacher on the MTT is a good indicator of success. On the other hand, when a large proportion of the students are of middle class background, the personal and social characteristics of the teacher appear more critical. Changes in teacher ability as measured by the MTT also seem to be related to the socioeconomic makeup of the community. The criterion measures involving supervisor ratings and the assessment of socioeconomic status are both of questionable validity. Nonetheless, the most obvious conclusion of the investigation was that the question of predicting teacher effectiveness is not to be simply answered by direct measures of obvious variables, but must be viewed as a complex interaction of several interrelated classes of variables. This is a good guiding observation for those interested in future teacher characteristic and effectiveness research.

## Teacher Classroom Activity

The studies of Torrance and Parent (1966), Peskin (1964), and Turner (1964) indicate promising approaches to identification of teacher attributes critical in predicting effectiveness. Even if similar studies identify other important teacher characteristic variables, however, such findings are of limited practical value as a source of theory on the teaching process. Teacher attributes correlated with effectiveness are useful primarily as predictive measures in the selection of those who will teach. Such factors as personality, intelligence, knowledge of mathematics, and experience are static elements of teaching measured prior to classroom activity. Measures of these variables indicate nothing about the way a teacher behaves in the classroom. To study teaching without looking at classroom activity is to overlook the dynamic aspect of the teaching process.

The classroom behavior of mathematics teachers has been popular topic of research for many years. However, the results of comparing various name methods of teaching--discovery, laboratory, lecture--are generally inconclusive and unreliable. Recent developments in instructional technology--television, computers, and teaching machines--offer, in addition to alternative teaching procedures, valuable tools for conducting research relevant to conventional classroom teaching.

The review in this section covers research on teacher activity in the classroom and is divided into five major areas: (1) methods of presenting mathematical ideas, (2) media for instruction, (3) teacher-pupil interaction, (4) class arrangement--including grouping, size, and team teaching, and

(5) instructional impact of certain curriculum development projects.

## *Teaching Methods*

*Discovery*. The "discovery method" is the pedagogical catch phrase in modern mathematics, and the most popular type of teaching methodology study is a comparison of some form of discovery teaching with conventional teaching. For example, Howitz (1965) taught one class of ninth grade general mathematics using a guided discovery method and contemporary mathematical content (NCTM *Experiences in Mathematical Discovery*) and another using traditional methods and content (Stein *Refresher Arithmetic*). After one school year of such procedure he found no significant differences in the achievement of the two groups, as measured by the *Sequential Tests of Education Progress in Mathematics* (STEP) and no significant difference in attitudes of the two groups toward mathematics. A specially designed posttest of achievement did show significant differences favoring the experimental class, but that is hardly surprising.

At the elementary school level, Fleckman (1966) investigated the effect of discovery teaching on learning of division. In a pair of studies involving a total of 246 fifth and sixth graders, she found that discovery teaching produced significantly better concept learning but no better skill learning than conventional procedures. The discovery method was defined and implemented by a combination of specially written materials and personal consultation with the investigator and the teachers.

Ballew (1965) and Price (1967) probed the impact of discovery teaching on critical thinking abilities of high school students. Ballew's study involved a single teacher and three classes, one a control class and two others using specially written discovery lessons for a period of eighteen weeks. Pre- and posttests of achievement (STEP) and critical thinking (*Watson-Glaser Critical Thinking Appraisal*) revealed no significant differences in achievement attributable to the teaching treatment, a significant difference in critical thinking favorable to one of the experimental classes, and no significant interaction between the teaching treatments and achievement, critical-thinking growth, intelligence, or aptitude.

Price designed three teaching treatments for high school general mathematics classes:

1. Control--Lecture and recitation.

2. Discovery--Special materials to promote discovery.
3. Transfer--Discovery plus special materials designed to promote transfer and analytic thinking methods.

Pre- and posttests of mathematics achievement, reasoning, inductive reasoning, and critical thinking plus a questionnaire on attitudes showed that discovery and transfer methods produced no significant difference in achievement, a greater increase in mathematical reasoning, and positive attitudes toward mathematics compared with the traditional methods. The transfer method produced a significant increase in critical thinking, which, in combination with Ballew's mixed result, indicates that discovery teaching can produce increases in critical thinking abilities of students only if special attention is paid to cultivating that ability and its transfer.

Two college level studies compared methods in the discovery/lecture families. Levine (1967) taught two classes of freshmen mathematical analysis using a theory-to-application approach in one and an experience-to-theory approach in the other. Scores on posttests and delayed posttests showed the experience-to-theory method superior for cultivating student ability to solve problems in mathematical analysis, ability to generalize, and ability to apply new generalizations. The question of interaction with specific student populations is raised by contradictory results from a similar study by Caruso (1966). Caruso taught concepts of abstract algebra to college freshmen by what were essentially rule-example and example-rule methods and found the former method more effective. Of course, differences in subject matter and criterion measures in the two studies also complicate comparison of findings.

In any classroom investigation of teaching styles, such as those cited above, two fundamental questions cast doubt on any results. Can a number of teachers clearly and consistently make their teaching behavior conform to a single discursively defined method? Does the name of a method, for example "discovery," mean the same thing to each of the reporting investigators?

One way to avoid the first difficulty is to present instruction through an impersonal medium, such as television or programmed instruction. Henderson and Rollins (1967) used the latter method to compare three types of discovery teaching. Analyzing the logic of concept formation, they derived three possible ways to teach concepts from geometry: (1) agreement--a series of positive instances only, (2) paired instances of agreement and difference, and (3) nonpaired instances of agreement and difference. They selected ten concepts from high school geometry to be taught to eighth graders by the various strategems. The strategems were written into teaching programs and administered to the students. Although Henderson and Rollins hypothesized that middle and low ability students would do better under Treatment 3, this was not borne out in the achievement testing. Apparently all three strategems are equally effective, suggesting that critical interactions between treatment and ability lie in more subtle variables.

The Henderson-Rollins procedure is one way to solve the problem of reliability in teaching methodology research. Of course, interpretation of such results as valid research on classroom teaching is questionable. Another approach would be detailed behavioral definition of the methods and use of classroom observational techniques (discussed later) to insure conformity to the methods. This, however, sidesteps the second question posed above, "Does 'discovery' refer to a well defined, commonly understood pattern of teacher behavior?" This is an issue still being vigorously debated in mathematics education.

The emerging point of view is that there are probably many forms of instruction that have discovery components. The proper direction of research then is to determine the differential impacts of each type rather than seek some single method best for all situations. Thus methodology research is shifting toward investigation of the complex interaction between content, teacher behavior, student aptitudes and student outcomes. The objective is to find the specific teacher behavior style most effective for specific mathematical topics and specific kinds of student populations. This emphasis, part of growing interest in individualization of instruction, is exemplified in ATI studies, studies of Aptitude-Treatment Interaction.

Aptitude-Treatment Interaction. The first challenge in ATI research, still in an exploratory stage, is to show that instructional treatments can be constructed that give evidence of aptitude-treatment interaction effects on relevant outcome measures. Kropp, Nelson, and King (1967) devised four packages of instructional materials for teaching elementary set concepts. Each package was designed to embody one of four treatments: (1) verbal-deductive, (2) verbal-inductive, (3) figural-deductive, or (4) figural-inductive. The material was presented to 400 elementary school students in two days. A battery of aptitude tests and a 24 item criterion test (containing equal numbers of verbal and figural items) were administered to test the following hypotheses:

1. Verbal ability will be more closely associated with success in verbally presented material than will figural ability; figural ability will be more closely associated with success in figurally presented material.
2. Measures of inductive reasoning will have significantly higher regression coefficients than measures of deductive reasoning when an inductive mode of presentation is used; deductive reasoning measures will have significantly higher regression coefficients when deductive presentation is made.

Test data showed that for heterogeneous groups the treatments were equally effective; the inductive-deductive interaction was significant at the .05 level in the direction hypothesized. The failure of verbal-figural interaction to appear might be attributable to inadequate distinction in the treatments, invalid aptitude measures, or irrelevance of the verbal-figural dichotomy in the chosen learning task.

Two doctoral studies derived from the same problem area (Behr, 1967, and J. Davis, 1967) examined the verbal-figural treatment dyad in other content areas with subjects at different age levels. Behr taught a one day programmed lesson in mod 7 arithmetic to 229 students in an elementary college mathematics course. Half the programs embodied a semantic-symbolic mode and half a figural-symbolic mode. He administered ability test in two sessions, treatments in one session, a learning test in the next session, and a retention test two weeks later. The major result was high correlation between semantic ability and success on the semantic treatment.

Davis (1967) tested the semantic-figural interaction in classes where the topics were finding the derivative of an algebraic expression and multiplying vectors. He observed no prominent interactions.

In another doctoral study of ATI effects, Becker (1967) designed two different kinds of instructional treatment for teaching summing of number series. Treatment A gave the learner the correct formula, in both verbal and symbolic form, for summing a particular series and then explained the structural relationship between formula and series. Treatment B broke the learning task into many steps leading to discovery of a formula. The treatments were written into programs and administered to Algebra I students, matched in pairs according to verbal and mathematical aptitude.

On tests of ability to recall terms, symbols, and formulas, ability to find the sum of n terms of a series, and ability to devise a formula for the nth term of a series, there was no evidence of significant aptitude-treatment interaction. As in the Kropp, Nelson, and King study, lack of interaction effect might be attributable to irrelevance of the treatment variables for the particular learning tasks.

Carry (1968) examined the interaction of spatial visualization and general reasoning abilities with two methods of teaching quadratic inequalities. He prepared programmed instructional treatments, characterized respectively as graphical and analytical. Students in nine geometry classes studied the programs for two days.

Results from an immediate recall learning test did not support the hypothesis of aptitude treatment interaction. A transfer test did indicate interaction, but this finding was confounded by low reliability of the transfer test.

The importance of ATI effects can be confirmed or denied only after broader exploration of possible treatment and aptitude variables. Use of programmed instruction has limitations as a research simulation of teaching methods, but the value of clearly defined, repeatable experiments seems to outweight these limitations right now. The brevity of teaching programs, while useful for experimental purposes, casts doubt on the application of findings to long-term teaching methodology.

Individualization. ATI studies are one important phase of the process of justifying and designing programs for appropriate individualization of instruction. One interesting study (Ebeid, 1964) explored the integration of an individualized component into normal classroom situations.

Using seventh and eighth grade classes studying SMSG material, Ebeid designed an experimental treatment in which one period a week (two during the second semester) was devoted to student self-selected study. Although pre- and posttests of achievement (STEP and special SMSG oriented test) and attitudes showed no significant differences between the experimental and control classes, all trends favored classes exposed to the individualization component.

Moody (1968) conducted a different kind of study that has a bearing on the question of individualization. He compared a student self-instructional procedure with a teacher-directed procedure in teaching concepts of nonmetric geometry to fifth graders. Comparing class means for 19 classes (6 self-instructional and 13 teacher-directed) he found achievement in the teacher-directed classes far greater than in the self-instructional classes. He also investigated correlation between teacher and student knowledge and found that the correlation was high on simple definitional type items but not on more complex items involving application or problem solving with basic concepts.

The differences noticed between the two methods of instruction might be due to weakness in the type of material presented to students for self-instruction or to lack of student experience in self-instruction. The results suggest, at least, caution in attempts to turn all learning over to student self-instruction.

*Disadvantaged Students*. The original impact of new curricula in school mathematics was in programs for capable students. Now there is growing interest in methods of teaching low achievers and disadvantaged students--a special kind of aptitude-treatment interaction problem. But the extent of this concern is not yet reflected in reported research. Four reported studies--Easterday (1964), Engel (1966), Castaneda (1967), and Jones (1968)--involved development and trial of new programs for low achieving and disadvantaged students.

In a study with more general implications for teaching low achievers, Wiebe (1966) compared three methods of using programmed instruction with such students. The treatments were (1) programmed instruction with immediate reinforcement, (2) half programmed instruction with immediate reinforcement and half teacher-directed instruction, and (3) half programmed instruction with delayed reinforcement and half teacher-directed instruction. Wiebe administered the treatments to 236 experimental subjects for five days. Pre- and posttests and a one week delayed posttest showed that Treatment 2 produced highest score and significant gain ($p < .05$). He concluded that low achievers apparently need some interaction with and instruction from a teacher to make most effective progress.

*Psychological Theory to Teaching Theory*. The recent reconstruction of the school mathematics curriculum has also led to a vigorous revival of interest in the psychological aspects of mathematics learning, led by Piaget, Bruner, Dienes, Suppes, and Gagne. Such a combination of curriculum and psychological research emphasis seems to reflect a belief that having determined the important mathematical ideas and how children learn these ideas, effective teaching procedure is a routine corollary. There is, however, a growing awareness that the matter is not so simple; translation of learning theory into meaningful guidelines for teaching is an important problem worthy of the best efforts of ingenious teachers. Evaluation of pedagogical theories derived from learning theories is an important emerging area of research.

The Cuisenaire method of teaching arithmetic is derived, at least in part, from the Piagetian theory that young children's thinking is limited to concepts with immediately available concrete referents. Gowder (1965) and Hollis (1964) each compared effects of teaching arithmetic by Cuisenaire and conventional methods and found the Cuisenaire method superior, particularly in concept learning.

In two other studies testing the pedagogical implications of Piagetian theory, Toney (1968) and Trueblood (1967) compared teaching methods in which students manipulated concrete instructional materials with methods in which the teacher simply demonstrated or described the manipulation. Results did not significantly support the student manipulation method (presumed to embody the proper psychological principles). The Toney study, however, involved a very small sample, and in both studies Piaget's theory has been oversimplified and superficially interpreted in the design of the two instructional treatments.

Woodward (1966) conducted an interesting experiment to test an idea of Ausubel that meaningful verbal learning can occur only when more inclusive relevant concepts exist and are readily available in the cognitive structure of the learner. The idea implies that presentation of an advance organizing concept prior to a verbal learning task will enhance learning. Woodward examined this hypothesis and a kind of converse which proposed that discovery learning would be more successful if overt postorganizers were presented after discovery.

He taught mod 11 arithmetic to a total of 44 college subjects using four methods--(1) preorganizers and discovery instruction, (2) preorganizers and verbal instruction, (3) postorganizers and discovery instruction, and (4) postorganizers and verbal instruction--each mediated by computer. A learning test (administered one day after instruction) and a transfer test (administered one week later) yielded no significant differences or interactions to support any of the hypotheses. An interesting next step in this direction would be comparison of those results with yet a third treatment, no pre- or postorganizers, reflecting Hendrix' admonition about premature verbalization and its effect on discovery learning.

In another sort of test of the advance organizers idea, Procter (1967) tested the hypothesis that student learning and class participation would increase significantly if (1) specific objectives were clarified for the student before presentation of the learning task, (2) effective feedback apparatus was available during the learning task, and (3) there was an objective relation between achievement of specified behavioral goals and course grades. Proctor derived these hypotheses from Gagne's theory that every learning task can be described in terms of desired behavioral objectives and analyzed into a number of prerequisite simpler objectives. His experimentation confirmed the learning aspect of the hypothesis, but level of student participation in class was not significantly affected by implementation of the stated procedures.

Yet another translation of psychological theory into teaching theory was reported by Farrell (1967). She attempted to devise an optimal approach to teaching high school geometry based on the cognitive activity of pattern centering--the process of sorting information into categories or patterns and analyzing constructed patterns to determine relations among them. From analysis of the subject matter and the learning characteristics of the intended students, she concluded such an approach would be appropriate. The crucial translation into actual teacher behavior prescriptions and testing the results were not a part of the study, but an interesting hypothesis was suggested.

## Media of Instruction

Programmed instruction is playing an important research role as a means of simulating classroom instructional methods under rigorously controlled, repeatable conditions. Programmed instruction, however, was originally conceived as a replacement for traditional classroom teaching--a tool which could offer pacing and feedback on an individual basis. Since mathematics has been a popular topic for developers of programmed material, there has been a great deal of research on the effectiveness of this new instructional medium. The most frequently debated and tested question has been the relative merits of programmed and teacher-directed instruction. Results of media comparison studies have failed to establish the superiority of programmed or conventional instruction. In fact, the evidence is sharply contradictory.

Zoll (1969) reviewed research in programmed instruction and found that in 13 such comparative studies three showed significant differences favoring programmed instruction, three showed significant differences favoring traditional instruction, and seven showed no significant differences. (See Devine, 1968, Kellems, 1965, or Dobyns, 1964, for example.)

Research in programmed instruction has thus, quite properly, moved in the direction of studies seeking the interactive effects between various types of programming, subject matter, and learner characteristics and studies of how programmed materials can be most effectively used in conjunction with standard teaching procedures. For example, Morgan (1965) and Callister (1965) examined stress, anxiety, and achievement interactions in programmed and conventional classes; and Wiebe (1966) tried to find the most effective combination of programmed and teacher-directed instruction with low achieving students.

In summary, it seems fair to conclude that, far from supplanting teacher classroom instruction, programming will probably prove to be an important instructional device to be integrated into a total instructional plan where most effective.

Television is another instructional medium that has come upon education offering great promise of improving instruction. As yet, the research evidence on methods and effectiveness of televised instruction in mathematics is slim, particularly in elementary and secondary schools.

The best known use of television in elementary school mathematics instruction is Patterns in Arithmetic (PIA), developed at the University of Wisconsin. PIA utilizes television lessons and coordinated teacher manuals and pupil exercise books in an arithmetic program for Grades 1-6. The course is now being used by over 135,000 students in eight states.

During the 1966-67 school year, the Wisconsin Research and Development Center for Cognitive Learning directed a summative, noncomparative evaluation of the PIA program in first- and third-grade classes of selected Wisconsin and Alabama Schools (Braswell and Romberg, 1969). Both standardized and specially designed achievement tests were used to measure arithmetic concept attainment, computational ability, and other special goals of PIA. The results were generally favorable to the televised program. Furthermore, opinion inventories showed that both teachers and students liked the televised course.

Multivariate analysis of variance was used to examine the effect of community size, state, and socioeconomic status on achievement in PIA. The only significant difference indicated that Wisconsin students did better on standardized computation tests than their Alabama counterparts. Trends favored high socioeconomic groups, particularly on the standardized tests of concept attainment.

In the secondary school, instruction by television is not at all common. Colleges are the main users of televised instruction, and this is more a reaction to exploding student populations in general mathematics courses than to careful appraisal of the merit in this medium.

Nazarian (1967) observed that most studies have shown television to be as effective as standard instructional techniques--at least on gross measures of achievement. He investigated the response to television of various ability groups. Using two classes each of high and low ability (measured by SAT score) college general mathematics students and an experimental instructional pattern of two television classes and one recitation per week, he found that a control, standard lecture method produced better achievement; high aptitude students did better in the experimental course, but low aptitude students did better in the control course. None of these differences was significant, however.

In another college level television experiment, Lane (1964) examined several ways of making up for the lack of teacher-pupil interaction when instruction is mediated by television. He found that a programmed booklet based on assigned problems was superior to a classroom help session and to a kinescope problem session.

From the evidence, or lack of evidence, in research, it is clear that the possibilities of television as a component in mathematics instruction have barely been tapped.

## Teacher-Pupil Interaction

If teaching methods research is ever to produce reliable results in classroom experimentation, it will be necessary to develop rigorous behavioral definitions of teaching styles and observational techniques that dependably and accurately measure conformity to the styles in question. Throughout educational research there is a growing interest in systematic, empirical study of teacher-pupil behaviors that shows promise of providing these necessary behavioral concepts and observational methods. Application of these techniques to study of mathematics teaching in particular has been limited, but promising.

Interaction Analysis. Flanders' system of Interaction Analysis gives a procedure for quantifying direct and indirect teacher influence in the classroom that has been used to test a variety of conjectures concerning the relationship between classroom climate and student achievement or attitudes. In a study of 16 mathematics classes, Flanders (1965) examined the following hypotheses: (1) Indirect teacher influence increases student learning when a student's perception of the goal is confused and ambiguous. (2) Direct teacher influence increases learning when a student's perception of the goal is clear and acceptable. (3) Direct teacher influence restricts learning when a student's perception of the goal is ambiguous.

All three hypotheses were supported. Successful teachers consistently exerted more indirect influence than direct influence. These successful teachers also showed a tendency to move from extreme indirect influence at the beginning of a unit of study, when goals were more ambiguous, to more direct influence as goals of the unit become clearer.

Interaction Analysis was also used by Torrance and Parent (1966), and it became part of an expanded instrument used to study pupil involvement and mathematical content in the "Five State Project" at the Minnesota National Laboratory.

The "Five State Project" was begun in 1961 as a field study of effectiveness of four experimental secondary school mathematics programs. One aspect of this evaluation (Wright, 1967) was an attempt to describe the effect of experimental curricula on patterns of teacher-pupil verbal interaction. Flanders devised a modification of his original system to determine involvement behaviors--the frequency and roles in which students are drawn into various aspects of classroom activity. A content oriented observational system derived from one of Wright and Proctor (1961) yielded data about the pattern of content behaviors--the relative emphasis of theoretical and conceptual mathematical activity in experimental and control classes.

Sixty-two observers used the involvement part of the observational system during the spring of 1964 and spring of 1965. The procedure used required the observer to note once every three seconds which type of involvement behavior occurred. The categories were:

Teacher -- (1) clarifying, encouraging, summarizing, (2) contacting, checking, (3) confronting, seeking, (4) soft or hard challenging, jolting, (5) informing, lecturing, (6) directing.

Student -- (7) receptive, passive, (8) independent, active, (9) curious, creative

General -- (10) silence, confusion, organization.

The accumulated data showed that pupils in both experimental and control classes played about the same role--passive or limited to highly controlled responses amounting to about 20 percent of all behaviors. The two significant differences showed that experimental teachers asked more confronting, seeking questions and their students responded more independently. However, this behavior constituted only four percent of all activity.

Twenty-five observers used the content part of the observational system during 1964 and 1965 also. The content behaviors were tallied every three seconds in one of three major categories:

(1) Fundamentals---consideration of the body of knowledge presently at command of the pupils. Subcategories--structure and techniques.

(2) Relations---consideration of new or broader concepts in mathematics. Subcategories--deductive, inductive, and statement.

(3) Applications---use or practice of a concept in specific problems, either in or out of mathematics. Subcategories--mathematical and other.

The data presented no clear pattern of frequency or sequencing among the content behaviors in either experimental or control classes. There was slight indication that class discussion in experimental programs paid more attention to theoretical matters than that in control classes without de-emphasizing basic problem-solving skills.

The lack of evidence of strong behavioral pattern differences between experimental and control classes might indicate that curricula alone do not exert the hoped for influence on classroom procedure. There were, however, four quite different curricula lumped together in the experimental classes. Furthermore, the content analysis system does not seem sensitive enough to fully and accurately record the structure of mathematical discourse.

Other Classroom Observation Schemes. Pate (1966) used a different kind of observational instrument to examine what he called Transactional Pattern Differences between classes studying SMSG and more traditional curricula. Contemporary mathematics programs generally place more emphasis on processes of discovery and inquiry and use more formal mathematical symbolism. Therefore, Pate sought empirical evidence about differences in teacher-pupil interaction in SMSG and traditional classes and the extent to which SMSG teachers implement the discovery and inquiry urged on them. He developed a composite observational instrument consising of a *Teacher Question Inventory*, an *Analysis of Patterns of Pupil Response*, and the *Prove Code* (Hughes et al., 1959). Observers recorded activity in forty classes, half using SMSG and half traditional programs, twice each.

The data corroborate findings in the involvement aspect of the Five State Project; that is, the predominant pattern of interaction was one in which recall and recognition questions were directed at individual students in a climate of control. Although there was some evidence that SMSG teachers used more divergent and analytic questions to elicit spontaneous and creative response, this behavior was only a small part of classroom activity.

Evidence from these two studies seems to indicate that suitably designed experimental curricula can have a noticeable influence on patterns of classroom interaction. Moreover, observational instruments themselves, consisting of behavioral concepts whose referents are identifiable in classroom activity, offer an instructional framework and language for meaningful efforts to change teacher behavior.

In a doctoral study, Buck (1967) used the Observation Schedule and Record (OScAR) to compare teaching behavior of intermediate school teachers with varying mathematical ability and classroom experience. He visited classes of teachers paired according to mathematics achievement and experience. Eight scales on the elementary mathematics version of OScAR reliably discriminated teachers of varying mathematics achievement, thus suggesting that knowledge of mathematics is a factor in choice of behavior.

Two other investigators, Stilwell (1967) and Fey (1969), developed special instruments for recording mathematically and pedagogically significant aspects of classroom verbal behavior. Stilwell devised 16 categories to cover possible behavior during problem-solving activity in geometry classes. Then using a three second time unit for observation, he examined problem-solving activity in classes of 12 teachers.

Data from those observations were used to develop a profile of classes for a "composite" teacher and to compare classes of individual teachers. Like Flanders, Stilwell combined teacher behaviors into two classes to get an E/D ratio indicating relative amounts of teacher talk encouraging and discouraging student involvement. He found that greater teaching experience had a significant positive correlation with amount of encouraging behavior. Another extremely interesting result was the fact that all teachers differed significantly ($p < .005$) from the composite teacher!

FEY

Fey (1969) developed a system for analyzing the pedagogical function, duration, content, mathematical activity, and logical purposes of each utterance in classroom discourse about mathematics. Following the lead of Bellack et al. (1966), he viewed teacher and student verbal communications as moves in a classroom language game and tried to discover the rules of this game as it was played in classes of five teachers participating in the Secondary School Mathematics Curriculum Improvement Study. The instrument gave evidence of being a sensitive indicator of the patterns of mathematical activity in secondary school classes.

In both the Stilwell and Fey studies the major objectives were to develop reliable and valid techniques for describing verbal behavior in mathematics classes and to generate hypotheses for further investigation. The value in this effort can be realized only if the studies are followed by experimental classroom research using the observational schemes.

## Teacher Organizational Responsibilities

The central responsibility of a teacher is active leadership of classroom learning activities. Thus research most often focuses on patterns and media of classroom instructional behavior. In preparation for teaching and during a given class session, however, the teacher is called upon to make a variety of procedural decisions involving grouping of students, class size, assignments, pacing, enrichment, and so on. A number of studies offer insight into most effective choices in these procedural decisions.

Grouping. As part of the attempt to present students with a program suited to their individual level of mathematical ability and achievement, homogeneous grouping has become a standard practice throughout the secondary school. Three reported experiments examined the effects of various ability grouping arrangements.

Stevenson (1966) investigated the interaction of ability, achievement, and grouping when 142 seventh grade students were grouped according to diagnostic pretests before each of ten units. He found that an average of sixty students changed groups at the end of a unit; there was no significant difference in achievement of arithmetic computation or application ability between those students who changed frequently and those who changed seldom; but those who changed seldom had a significantly higher arithmetic concept gain.

A similar flexible ability grouping procedure was tested by Willcutt (1967). He failed to find significant differences in achievement between experimental and control classes, but a significant difference in attitudes favored the flexible grouping procedure.

Campbell (1964) compared whole class instruction and a within class grouping procedure in seventh grade arithmetic instruction. In each of the four experimental classes, achievement tests and teacher judgment were used to determine three subgroups. Each subgroup was placed in an area of the classroom and given group instruction by the teacher--completing seatwork while the other subgroups were instructed. Tests revealed no significant differences between the experimental and control class achievement, but the trend favored the whole class method of instruction. Teacher attitudes seemed to favor the grouping procedure.

The most thorough and definitive study of ability grouping in mathematics instruction was reported by Neill (1966). As part of a large scale study (Goldberg, Passow, and Justman, 1966), Neill's investigation compared the effects of enrichment and acceleration on achievement of 1477 academically talented junior high school students. Using the Lorge-Thorndike Verbal Intelligence Test and STEP reading and mathematics tests to determine initial status, and using three different criterion tests, he found that a contemporary mathematics program taught under acceleration leads to greater achievement gains than an accelerated traditional program or an enriched program of either traditional or contemporary content.

Neill also checked for interaction between various teacher and student attributes and achievement. He found that teacher characteristics contributed substantially less to differences in achievement than student attributes, the main teacher predictor being length of academic preparation. Student attributes closely associated with achievement were intelligence, initial reading and mathematics ability, socioeconomic status, attitudes toward mathematics, and self-appraisal. There was no significant interaction of pupil and teacher sex, but classes taught by men were more successful (a result confirmed in the International Study of Mathematics Achievement).

Team Teaching. An organization of instructional responsibility that promised to make most effective use of the talents of individual teachers, team teaching has not gained wide acceptance in mathematics. There has hardly been enough research into the procedure to allow fair appraisal of its value.

Paige (1967) conducted a small study (one experimental and one control class) to compare team and traditional teaching of junior high school mathematics. Students in the traditional group were matched in pairs by sex, interest, mathematical achievement, and previous grades with students in the team-taught group. Three criterion measures--a posttest, a delayed posttest, and a relearning test--failed to give evidence of significant differences between the two treatments.

In another less tightly designed study, Bhushan, Jeffryes, and Nakamura (1968) tried to reduce teacher load (in terms of classroom contact hours) by a procedure combining large group (69 students) lecture instruction, small group (23 students) discussion and problem sessions, and independent or tutorial study. The variety of instructional procedures was made possible by flexible scheduling procedures. Achievement testing showed no significant differences between experimental and control classes. Thus the procedure apparently succeeded in reducing teacher class contact time without reducing learning.

Class Size. The problem of optimal class size, like that of television effectiveness, has been faced primarily by college teachers of mathematics. Kerce (1965) examined the interaction between class size and teaching method. Small (13 students) and large (54 students) classes in freshman college mathematics were taught by discussion, lecture-laboratory, lecture, and mixed methods. There were no significant differences in the treatments or class size comparisons. Surprisingly, however, attitude measures favored large class procedures.

C. Moore (1967) compared two procedures in freshman college mathematics--a combination of large lecture and small study classes and the traditional lecture recitation method. He found no significant differences in achievement; attitudes and opinions favored the experimental treatment; and there was a decline in debilitating anxiety under the same procedure.

In a study of large group instruction at the high school level, Madden (1966) found achievement in ninth grade general mathematics was significantly greater in large classes (70-85 students) than in normal size classes (25-40 students). In this study, however, and in all other comparisons purportedly testing only the effects of class size, the more important question seems to be, "What methods of classroom teaching are most effective with groups of a particular size, students of particular ability levels, particular mathematical subject matter?" Moreover, to get a true measure of the effectiveness of various procedures, it will be necessary to apply criterion tests that probe beyond the gross measures of achievement and attitudes.

## UICSM and The Madison Project

With experimental research on mathematics teaching so consistently inconclusive, teachers have been forced to look elsewhere for fresh ideas about their instructional tasks. The Madison Project, under direction of Robert Davis, and the University of Illinois Committee on School Mathematics (UICSM), under direction of Max Beberman, have both had an important impact on the way teaching is in mathematics today, even though both are primarily curriculum development projects.

UICSM has supported its new curricular material with creative teaching ideas and made the important effort to convey these ideas to teachers. The most recent UICSM product is a junior high school program for low achievers. It consists of two courses--Motion Geometry (1969a) and Stretchers and Shrinkers (1969b)--written to especially appeal to low achieving students and accompanied by teaching guidelines found effective in clinical tests.

The Madison Project has more clearly concerned itself with producing new teaching ideas, but even less than UICSM with formal experimentation. Davis and his colleagues are clinicians drawing on the insight and experience of psychologists, mathematicians, sociologists, anthropologists, and even occasionally a musician to develop novel approaches to mathematics instruction. After formulating the instructional strategies in classroom exploration, they try to analyze the teaching processes to determine those features crucial for success. These observations are then reported as clinical contributions to a theory of mathematics instruction.

In one thorough report Davis (1964) discussed the Madison Project's viewpoint on (1) the kinds of mathematical experiences that should be provided for children, (2) criteria for selecting experiences, (3) teaching as a flexibly programmed discussion sequence, (4) classroom reinforcement strategies, (5) student freedom, motivation, and learning, and (6) the interpretation of Piaget's concepts of assimilation and accommodation in teaching practice. But these findings are presented as the observations of an astute classroom teacher; how do they qualify as research? The Madison Project does not report controlled experimental studies of teaching in the traditional format. Instead, Davis (1967) argues eloquently and persuasively that his evidence is the sequence of films of Madison Project classes in action and that these films should be accepted as bona fide research evidence. These films and other project work do suggest many ideas for further experimental research and practical suggestions for classroom teachers, so their acceptance as contributions to understanding mathematics teaching--and thus status as research--is undeniable. Anyone interested in building theory on instruction in mathematics would be remiss in overlooking these informal but highly insightful exploratory studies.

## Teacher Education

The quality and originality of recent research into characteristics and behavior of effective teachers are not at all evident in studies of pre- and inservice education. Shortly after the new curricula became widely used, recommendations were presented by CUPM and the Cambridge Conference for improvement of teacher education in mathematics. Several studies have explored the extent of implementation of these recommendations and, particularly in elementary teacher education, others have examined the impact of these changes on the competence and attitudes of newly educated teachers. Only a little research offers fresh insight into the phase of teacher education involving preparation for classroom mathematics teaching behavior. The best known programs of this type are interdisciplinary in scope.

### Secondary Teacher Education

The major recommendation for change in the education of secondary school mathematics teachers has come from the Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America (1960, 1965, 1966). Fisher (1966) surveyed the extent of change effected by these recommendations and found noticeable movement in the suggested direction. Only preparation in geometry and probability and statistics remained weak.

The National Science Foundation, through its extensive program of inservice, summer, and academic year institutes, has been a major force in reeducating inservice teachers. The notable result of two different evaluations of Academic Year Institute programs (Irby, 1967, and Wilson, 1966) is the fact that over 40 percent of all participants use the institute as a stepping stone to college teaching positions.

As part of a larger cross national study of teacher education, Wiersma (1967) compared the academic achievement of mathematics majors preparing to teach in the secondary schools. Prospective teachers were examined one month before completion of their teacher preparatory programs in the United States, England and Wales, and Scotland. On six achievement measures, the United States students scored lowest on four (including mathematics), middle on one, and highest on biological sciences. A variety of explanations were offered for these results, suggesting hypotheses for experimental teacher education studies.

Moser (1965) and Steinen (1966) each investigated ways of improving student teaching experience by providing increased feedback on performance. Using a technique employed widely in other disciplines, Moser made tape-recordings and an interaction analysis of observations in order to carry out an objective analysis of the teacher performance. He tried to determine whether patterns of teacher behavior could be identified and whether the feedback procedure had any influence on teacher behavior patterns.

After each observation, the supervisor and the teacher played back the tape and discussed the lesson within the framework of interaction analysis. Although teachers seemed to value the information provided by the interaction analysis, it appeared that they quickly sought a personal style of teaching and adhered to it rigidly. Of interest in connection with the Stilwell study is the fact that popular teachers did not have similar styles as measured by the interaction analysis.

Steinen used three different sources of feedback--the student teachers themselves (they taught the same topic to classes one or two days out of phase), fellow student teachers (the students were paired in teaching teams), and pupils (they completed anonymous questionnaires). Although the investigation was primarily exploratory, evidence seemed to indicate all three experimental procedures significantly more successful than normal procedures.

## Elementary Teacher Education

Increased preparation in mathematics is one of the most noticeable changes in recent elementary teacher education. The assumption behind this change is that improved competence in mathematics will lead to better attitudes toward the subject and more effective teaching.

Evidence on the question of attitudinal change is mixed, with an apparent trend for increase in understanding of mathematics to be accompanied by more positive attitudes. McLeod (1965) found no correlation between achievement on a concepts test and attitudes, arithmetic skills, years of experience or grade level of a group of inservice teachers. Purcell (1964) found attitudes, understanding of concepts, and grades in a methods course not significantly related. Gee (1965) and Todd (1966), however, did detect significant positive shifts in attitudes accompanying growth in mathematical understanding. Reys (1966) noticed some trend in the same direction.

There is almost no evidence to support or deny claims of correlation between teacher knowledge of mathematics and classroom effectiveness. Haukebo (1967) tested the frequently stated assumption that study of numeration systems enhances and reinforces understanding of decimal system, and the hypothesis was not supported.

What kind of mathematics instruction is most effective for elementary teachers? Since the crucial criterion--effectiveness of the prospective teacher in the classroom--is so difficult to appraise, mathematics achievement and attitudes are normally offered as indicators. A variety of experimental instructional arrangements have been reported.

Dutton (1966) found individualized instruction could be achieved by using programmed instructional materials. Foley (1965) found no significant differences between the achievement of students in a large class (203 students) and those in a normal size class. He also detected no differences in attitudes.

Northey (1967) evaluated three methods of teaching elementary teachers determined by varying time allotments to lecture and discussion activity. Using computation skills, attitudes, concepts learning and retention, and unit achievement tests as criteria, he found no significant differences overall according to the treatments, but a variety of interactions between treatment and individual criterion measures and ability.

Bassler (1966) compared teaching methods determined by two types of reinforcement (immediate and delayed) and two types of problem exercises (physical world and mathematical settings) with standard developmental teaching. He found that neither type of problem-generated teaching was as effective as normal teaching and there were no interactions of student ability and treatment. He reported no reinforcement effects.

Henkelman et al. (1967) reported use of a behavioral objectives and task analysis approach to develop an experimental inservice teacher education program. Desired behavioral outcomes were analyzed into hierarchies of simple components and an instructional sequence was devised that would most efficiently lead to student acquisition of the desired behaviors. No data indicating comparative achievement was included in the report.

Television is beginning to play an important role in the immense task of pre- and inservice mathematics education for elementary teachers, but the tests of effectiveness for this medium are inconclusive.

Pehtel (1968) compared three methods of closed circuit television: (1) straight television, (2) television and discussion, and (3) lecture without television. He found no significant differences between the three treatments but a significant interaction between treatments and ability of students. High ability students did better with the mixed television and discussion method than with the nontelevision method.

Dwight et al. (1966) reported a comparison of mathematics instruction for elementary teachers by television and by standard classroom methods. On a battery of eight pre- and posttests a regular class averaged better than the television class, and the difference increased with time.

In doctoral studies, Green (1967) and Byrkit (1968) evaluated aspects of a multimedia system for teaching mathematics to elementary teachers. Lessons in the system Modern Mathematics for the Elementary School Teacher by Green and Kalin consist of a pretape worksheet followed by discussion with an instructor, a televised lecture, posttelevision worksheet, homework, and a summary.

Green tested the system by comparing its effectiveness and that of regular instruction with 142 inservice elementary teachers. Achievement and retention tests revealed no significant differences between the treatments. Then Byrkit substituted a programmed text unit for the initial worksheet and teacher discussion and compared effectiveness of the revised system and another approach using only the audio portion of the television tapes. He found no differences between these two treatments.

Stochl (1963) compared the effectiveness of practice teaching observations made in person with those done on videotapes. Using the methods course final exam and a special film test on teaching mathematics as criterion measures, he found no significant differences between the treatments.

In a noncomparative study, Mills and Kopstein (1965) evaluated an inservice course in mathematics for elementary teachers in which viewing of 30 minute telecasts was followed by seminar discussions. The program consisted of 15 lessons and was viewed at 12 pilot centers in four regional centers.

Pre- and posttests of arithmetic skill, structure of the number systems, and arithmetic insight revealed that the 192 teachers who completed the entire televised course did not benefit significantly. On the other hand, White (1963) found that a different television course in arithmetic did lead to highly positive attitudinal changes in a group of 92 prospective elementary teachers.

In both of the preceding studies, the results are only appraisals of particular televised courses, not evaluations of television itself. The appropriate question, and the most promising direction for future research, is "What use of television is most appropriate for a given subject and student population?"

## Summary

Recent research in mathematics teaching has produced no major breakthrough in the search for personal characteristics, education, or classroom behavior of effective teachers. However, several promising trends are emerging in the focus and techniques of research.

First, there is a growing realization that effective teaching is the result of a complex interaction between teacher ability, attitudes, and behavior, student aptitudes and attitudes, and the structure inherent in mathematical topics. The traditional search for a simple profile of a composite "good teacher" is giving way to investigations that ask "What kinds of teaching style and subject matter organization are most effective for teaching a particular topic to some particular student population?"

Second, there is a realization that student achievement on some standardized test is a grossly inadequate measure for teaching success. More comprehensive diagnostic assessments of student outcomes must be developed and used.

Third, several previously uninvestigated classes of teaching variables have been uncovered by exploratory studies. The creative thinking ability of teachers has been suggested as an important determinant of success. Classroom observational techniques offer powerful new methods in the study of teacher-pupil interaction variables. The most effective instructional use of television, programmed materials, and computers is as yet undetermined. And the pedagogical implications of recent developments in psychology are just beginning to be explored.

Fourth, the wide range of research results bearing on the activity of classroom teaching must be integrated by a theory of mathematics teaching and translated into programs of teacher education. Although these problems are receiving attention elsewhere in education, little has been done to consider the particular implications for mathematics.

Results of research on mathematics teaching have often been unreliable and inconclusive, but there is promise of more rigorous and creative work in the near future.

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# PIAGETIAN STUDIES AND MATHEMATICS LEARNING[1]

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Although most of the research and direct applications from Piagetian theory currently focus on preschool and elementary school situations, the insights into human learning and cognitive development that can be derived from a Piagetian point of view are equally applicable to secondary school situations and particularly to mathematics teaching and learning. The present paper, an attempt to substantiate the preceding opinion, is divided into six sections, some of which may be of greater interest to a particular reader than others. The first section consists of a brief description of the main features of Piaget's Theory of Intellectual Development. The second section summarizes Piaget's views on Factors Influencing Intellectual Development, a topic less frequently discussed in the literature than the theory of stages of development itself. The third section is devoted to conveying the nature of some of the Elementary School Piaget-Related Research that has been completed and that seems most relevant to arithmetic teaching and learning. The fourth section describes a few of the very few Piagetian Studies in Secondary School Settings that have been carried out. Then there is a section on Educational Implications from Piagetian theory and research, including particular references to secondary school mathematics learning and teaching situations. Finally, some specific Sample Mathematics Teaching Approaches that fit a Piagetian frame of reference very well are characterized.

Piaget's theory of intellectual development clearly describes true or living learning as originating from the child and his own interests and drives. Too often the child comes to view "school learning" as a process of mastering this or that skill " for the teacher" or "to get by," or "to beat the system." Whereas in real life the child learns from active interaction with his environment, doing things that are vitially interesting to him and for his own welfare. He explores, experiments, and modifies his behaviour and conceptions of the world by means of basic self-fulfilling drives. Can a child be led to operate this way in the world of mathematics? Some clues as to how this might be done can be found in Piagetian theory.

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## Piaget's Theory of Intellectual Development

A basic notion in the theory developed by Piaget and his colleagues in Geneva is that of a *schema*, a cognitive structure which has reference to a class of similar action sequences from past experience (Flavell, 1963, pp. 52-53). A schema can be thought of as the structure common to all those acts that an individual considers to be equivalent. For example, if a child's experience has led him to believe that putting three objects with four objects results in a group of seven objects, he possesses a basic schema that will enable him to understand that "3 + 4 = 7."

Any problematic situation requiring behaviour which is already generally represented in the child's mind is handled by being assimilated to the schema. Learning that "3 + 4 = 7" is assimilated to the knowledge that 3 objects and 4 objects make 7 objects. Furthermore, a child with such an operational schema is in a position to understand that three hundred plus four hundred equals seven hundred without ever having to count out that many beads or matchsticks (Skemp, 1958, p. 70). As another example of assimilation, suppose that a person has just arrived in a large city with which he is completely unfamiliar. He wants to walk from where he is to another location in the city. Chances are that he might purchase a city map, find out which direction is north, and proceed to plan a route for his walk that will take him to his destination. Such a sequence of actions would, in all likelihood, involve an assimilation of the situation in the strange city to what he would have done to proceed to a location in an unfamiliar part of his home city.

On the other hand, if the individual possesses no completely relevant schema, new behavioural sequences must be built through experimentation or instruction, or both, to enable existing schemata to accommodate to the new situation. For example, a considerable amount of relevant experience and training would likely be necessary to enable a person who has lived in cities all his life to find his way in a wilderness region, let alone survive, even though the city dweller might be especially proficient at finding his way in urban areas. He would need new relevant experiences to produce modifications of his schemata to accommodate to the contingencies that might arise out of the wilderness setting. Adaptation of an individual to his environment results from the interplay of assimilation and accommodation.

Another basic idea in Piaget's description of the development of knowledge is that of an *operation*. An operation is an interiorized action which can modify objects of knowledge. For example, an operation could consist of constructing a classification of objects, of putting things in a series, of counting, of measuring, of imagining an object being modified from one shape to another.

> To know an object is to act on it. To know is to modify, to transform the object, and to understand the process of this transformation, and as a consequence to understand the way the object is constructed (Piaget, 1964, p. 176).

An operation is reversible in the sense that it can take place in both directions (e.g., joining and separating). The attainment of reversibility requires more than an ability to actually undo an observed transformation, in that the individual must anticipate in thought a return to the state prior to the transformation (Inhelder, 1962, p. 35).

In the child's development of operational structures, the basis of knowledge, Piaget has distinguished four main stages: a sensory-motor, pre-verbal stage extending through approximately the first eighteen months of life; a pre-operational stage extending from about eighteen months to about seven years; a concrete operations stage from about seven years to about eleven or twelve years; and a formal operations stage which begins at about eleven or twelve years of age. Although the order of succession of stages is constant, and is the important notion, the chronological ages corresponding to the stages vary a great deal from culture to culture and individual to individual (Inhelder and Piaget, 1958, pp. 177-178; Flavell, 1963, p. 86; Inhelder, 1962, pp. 26-27).

During the sensory-motor stage (first 18 months) is developed the practical knowledge on which later representational knowledge is built. For example, the schema of a permanent object is constructed so that, toward the end of this stage, the infant will try to find a previously seen object when it is outside his perceptual field whereas he would not have done so in the first few months. Consequently, a notion of practical or sensory-motor space is constructed along with notions of temporal succession and elementary sensory-motor causality (Piaget, 1964, p. 177).

The pre-operational stage (up to about 7 years) is marked by the beginnings of language, of the symbolic function, and consequently, of thought or representation. Reconstruction of all that was developed on the sensory-motor level must occur at the level of representational thought. Throughout this stage of pre-operational representation, there is no evidence of conservation (perception of the basic elements of a situation that remain constant under certain transformations -- the psychological criterion of the attainment of reversible operations). For example, given two equal balls of plasticine and asked to roll one of them into a sausage shape, the child will assert that there is more or less substance in the sausage than in the ball, depending on whether he focuses on the increase in length or the decrease in diameter. He cannot seem to relate different aspects or dimensions to one another, and he tends to be deceived by his perceptions. He cannot mentally imagine the sausage being re-rolled into a ball of the original size. His concrete thought processes are irreversible (Piaget, 1964, p. 177; Inhelder, 1962, pp. 25-26).

In the stage of concrete operations, extending from about age seven to about age eleven, a thought structure, not yet separated from its concrete context, is formed. The first operations appear along with systems of operations that can be carried out simultaneously -- in contrast with sensory-motor actions which are carried out only in succession. The operations are concrete in the sense that they operate on real objects. Examples of operations developed in this stage are those of classification, of ordering, and of construction of the notion of number, as well as operations of spatial and temporal nature, operations of the elementary logic of classes and relations, and operations of elementary mathematics, geometry, and physics. The yet incomplete systems of operation are characterized by two forms of reversibility: negation, in which a perceived change is seen to be cancelled by its corresponding negative thought operation; and reciprocity, in which, for example, "being a foreigner" is seen as a reciprocal relationship (Piaget, 1964, p. 177; Inhelder, 1962, p. 26). Prior to this level of development the child has difficulty seeing himself as a foreigner to people from other lands. Similarly, a boy might assert that another boy is his brother and yet deny that the second boy has a brother. Younger children are basically egocentric and have difficulty dissociating themselves from any given situation.

The fourth stage, that of formal operations, beginning on the average at about eleven or twelve years of age, is characterized by the development of formal, abstract thought operations with which the adolescent can reason in terms of hypotheses and not only in terms of objects. Prior to this level of development the child thinks concretely rather than reflectively, dealing with each problem in isolation and not integrating his solutions by means of any general theories from which he could abstract a common principle. In contrast, the adolescent is most interested in theoretical problems and in constructing theoretical systems (Piaget, 1968, p. 61). The adolescent can identify all possible factors relevant to a problem under investigation, and he can form all possible combinations of these factors, one at a time, two at a time, three at a time, and so on. He can form hypotheses, construct experiments to test the hypotheses against reality, and draw conclusions from his findings. He need no longer confine his attention to what is real but can consider hypotheses that may or may not be true and work out what would follow if they were true. That is to say, in addition to considering what is, he can consider what might be. The hypothetico-deductive procedures of mathematics and science have become open to him (Piaget, 1964, pp. 177-178; Inhelder, 1962, pp. 21, 27-28; Adler, 1966, p. 579; Berlyne, 1957, pp. 8-10).

An example of formal operational thought is that carried on by the adolescent in coping with a problem in which he is given five bottles of colourless liquid, of which the first, third and fifth combine to form a brownish colour, the second is neutral, and fourth bleaches out colour. The problem is to find out how to produce the coloured solution. The adolescent discovers the combinatorial method, reasoning through the construction of a table of all possible combinations and experimentally determining the effect of each factor. This type of reasoning is beyond younger children (Inhelder, 1962, p. 27).

## Factors Influencing Intellectual Development

According to Piaget, the development from one set of mental structures to another is explained by the operation of four factors: maturation, experience, social transmission, and equilibration. He states that none of these is sufficient by itself to account for the preceding descriptions of mental development, but he considers the fourth, equilibration or self-regulation, to be the fundamental factor (Piaget, 1964, p. 178).

Even though maturation of the nervous system plays an indispensable role in development, it does not explain everything because the average age at which each of the various stages occur (but not the order of occurrence) varies widely from society to society (Piaget, 1964, p. 178).

Experience of objects, of physical reality, is also a basic factor in the development of cognitive structure, but it does not explain everything. For example, some of the concepts which appear at the stage of concrete operations cannot be drawn from experience alone. Consider the fact that a child becomes cognizant of conservation of substance at approximately age eight, but he does not assert that weight or volume is conserved until some time later. Weight and volume can be perceived directly, but how can the amount of substance be considered without notions of weight and volume? The child comes to understand that when there is a transformation of the shape of a quantity of plasticene, for instance, something must be conserved because the transformation can be reversed so that the plasticene can be returned to its original condition. Since it is not yet the weight and not yet the volume that is seen to be conserved, the notion of conservation of substance is simply a logical necessity -- no experience can show the child at this level that there is the same amount of substance. Furthermore, experience is of two psychologically distinct kinds: physical experience and logical-mathematical experience. Physical experience conforms to our usual notions of acting on objects and gaining some knowledge about the objects through the process of abstraction. Logical-mathematical experience, on the other hand, is drawn from the actions effected on the objects. For example, a child discovers that, no matter how he arranges a certain set of pebbles and no matter in what direction he counts them, he always has the same number. To make a sum and to order the pebbles, action is necessary. The child has discovered that the action of putting together (summing) is independent of the action of ordering -- this is a property of the actions, not of the pebbles. Herein lie the beginnings of mathematical deduction, which are further developed by the interiorization of the actions so that they can be combined without the need of pebbles. Before the formal operations stage, the coordination of such actions requires the support of concrete material, but it later leads to logical-mathematical structures in which operations are combined through the use of symbols and earlier logical-mathematical structures are used as a point of departure in thinking about new combinations. The source of logic lies in the coordination of such actions are joining together and ordering. Logical-mathematical experience, and experience of the individual's actions, not an experience of the objects themselves, is necessary before there can be operations (Piaget, 1964, pp. 178-180).

Social transmission, linguistic or educational, is a third basic factor. As an example of societal effects on development, consider Piaget's observation that the emergence of formal thinking corresponds to the age at which society expects the child to begin assuming adult roles. Not the onset of puberty, but the pressure to assume adult roles is the distinctive feature of adolescence in modern civilizations (Inhelder and Piaget, 1958, pp. 335-336). However, in order for a child to receive information from society he must have a structure that enables him to assimilate the information. Consequently, social transmission by itself is not adequate to explain development. Ordinarily a five-year-old, for example, cannot be taught higher mathematics because he does not yet have the structures that would enable him to understand (Piaget, 1964, p. 180).

Equilibration, the fourth factor, serves to relate the other three factors. An individual engaged in the act of knowing is led to react to compensate for external disturbances so that a state of equilibrium can be reached. The process of equilibration leads to operational reversibility, which is characterized by an equilibrated system in which a transformation in one direction is compensated for by a transformation in the other direction. This active process of self-regulation embodies the concept of feedback from the individual's interactions with his environment, and it takes the form of a succession of levels of equilibrium. A system is in equilibrium when a disturbance which modifies the state of the system has its counterpart in a spontaneous action which compensates for it. Levels of equilibrium can be identified according to the probability of the occurrence of various possible forms of compensation. Laws of equilibrium determine, at each stage of development, the best forms of adaptation compatible with maturation, experience, and influence of the social milieu. For instance, the pre-operational child can only cope with one dimension at a time and is led to assert non-conservation of a substance whose perceived form is altered, whereas a child in the concrete operations stage is able to take account of compensating changes in dimension (focusing on the transformation and not on the final configuration) to arrive at the notion of conservation (Piaget, 1964, pp. 181-182; Inhelder and Piaget, 1958, pp. 368-369).

In the formation of the ability to conserve quantity, for example, the following stages of strategy are distinguished: (1) considering one dimension to the neglect of others is the most probable strategy in the beginning, (2) emphasizing the second dimension becomes the most likely as a result of employment of the first strategy, and (3) oscillating between observed compensating changes in the different dimensions becomes most likely as a result of the preceding strategies. The process of equilibration starts at the level of self-regulation and sensory-motor feedback and leads to operational reversibility and intelligent thought at higher levels of development. Every new problem produces disequilibrium which is recognizable by the dominant types of errors made in coping with the situation. A solution is often arrived at by synthesizing a new operation from formerly distinct operations to produce a new state of equilibrium. Consider, for example, the derivation of the concept of ordinal number from the process of cardination and the action of ordering. The "second" element in a row is the one that has one predecessor, and the "third" element is the one with two predecessors, and so on (Piaget, 1961, pp. 279-281).

Piaget's views on the development of cognitive processes are well known, but what can be said about his view of the learning process? He maintains that the learning of any logical-mathematical structure can be accomplished only if the teacher can build the structure to be learned from simpler, more elementary logical structures. This view is derived from the notion that logical structures are not the result of direct physical experience. They can be grasped only through the function of internal equilibration or self-regulation in coping with the characteristics of various actions. By way of example, Piaget and Inhelder have led five-year-olds to grasp conservation of number by getting them to drop beads simultaneously into a glass they can see and one they cannot. Usually children cannot grasp conservation of number until seven or eight if they are presented situations in which a one-to-one correspondence between two rows of objects is set up and then one row is spread out. The former (hidden glass) structure is analogous to the latter, but it is embodied in a simpler situation. It has been found that children can generalize from the simpler situation to grasp the concept in the more difficult setting. Learning of a complex structure is possible if such learning is based on natural development from and perception of relationships to more simple structures. In short, the learning of structures seems to obey laws similar to those governing the natural development of these structures. Learning is subordinated to development. Learning is only effective if it is lasting, if it can be generalized to new situations, and if the learner's operation level is raised. Naturally developed cognitive structures satisfy these criteria, and "learned" structures should satisfy nothing less. Furthermore, learning is possible only when there is active assimilation on the part of the learner, assimilation in the sense of integration of reality into cognitive structures (Piaget, 1964, pp. 182-185).

Thus far some of the theoretical ideas underlying Piaget's experiments have been reviewed and now an attempt is made to indicate the nature of some of the Piaget-related research that has been carried on at the preschool and elementary school level and that seems most relevant to arithmetic teaching and learning. Following that, descriptions are given of a few of the very few Piaget-related studies that have been carried on in secondary school situations. The balance of the paper is then devoted to discussions of educational implications from Piaget's theory that seem most relevant to mathematics teaching and learning.

## Elementary School Piaget-Related Research

Piaget's The Child's Conception of Number (1952) has inspired many replication studies, which have, on the whole, confirmed his findings that the child's ability to conserve number (i.e., to understand that the number of elements in a group does not change no matter how they are rearranged) is arrived at gradually and that the child passes through three stages in attaining number conservation. First, there is a period of nonconservation in which the child's judgment regarding the equivalence of two sets is dominated by perceptual impressions.

Following this is a transitional stage in which the child vacillates between conserving and non-conserving, depending on the nature and extent of the observed transformations. Finally, a stage is reached in which the child's conceptions of number become stable -- he is no longer deceived by appearances. He sees that the number of elements in a group does not change no matter how they are spaced out or rearranged. He has attained conservation of number (Almy, 1966, pp. 22-34).

For example, a child of about four years of age who is in the nonconservation stage will typically approach the problem of finding enough eggs to fill a row of seven egg cups by making an equally long row using perhaps only four eggs. When he is asked to put one egg in each cup he is surprised that there are not enough eggs. Similarly, if he is presented with a row of twelve eggs having the same length as a row of seven egg cups, he will typically assert that all twelve eggs will go into the cups. Even after being led to match a set of egg cups one-for-one with a set of eggs, he doubts that the two sets are equivalent when they are no longer lined up. In this early period, number situations are responded to on a purely perceptual basis even if the child can count (Almy, 1966, p. 27).

In the second, transitional stage the child has no difficulty setting up a one-to-one correspondence between two sets of objects, but he cannot maintain it when the arrangement of the sets is changed (Almy, 1966, p. 27).

In the third, conservation stage the child discovers that any change in the spatial arrangement of the objects can be corrected by an inverse operation. The child retains access to the information he derived from his observation of a one-to-one relationship and he can use this information in spite of any perceptual discrepancy (Almy, 1966, p. 27).

Piaget's proposition that discrimination, seriation, and numeration follow in that order in the child's development of number concepts has been borne out in a study by David Elkind (1968, pp. 56-75) involving ninety children ranging in age from four to six. Following Piaget's procedures, he administered three tests involving two sets of nine size-graded sticks (the second set was intermediate in size to the first set). In the first test, a discrimination test, the child was presented with one set of nine sticks in disarray on a table. He was given a score of one point for successfully being able to do each of the following things: find the smallest, find the largest, find the smallest after the sticks had been arranged so that the smallest appeared larger than the other sticks, and find the largest after it had been disguised (four points altogether). The second test, a seriation test, involved presenting the child with nine sticks disarrayed and asking the child to make a stairway just like one that the experimenter constructed and then dismantled. If the child was not able to do this with nine, five were removed. If he was successful in making a stairway with four sticks he received one point. If not, he was not tested further. If successful, he was asked to build a stairway with seven sticks (for one more point). If successful again, he was asked to build a stairway with nine sticks (one point). Finally, if he succeeded with nine, five more sticks selected at random from the second set of sticks were brought out and the child was asked if he could put them where they belonged (for one more point). The third test, a numeration test, involved presenting the child with an intact stairway of nine sticks and asking him to count the number of sticks (one point). Then a doll was placed on the first stair and the child was asked: "How many stairs does the doll have to climb to get on this stair?" If the child was able to answer correctly when each stair in the stairway was pointed to in succession, he was given one more point. A further point was given if the child could answer the preceding question when the fourth stair was pointed to and when the seventh stair was pointed to. Finally, the sticks were mixed and again the fourth and then the seventh stairs were pointed to and the child was again asked how many stairs the doll would have had to climb (for one point). The purpose of giving the numeration test was to determine whether the child could coordinate an ordinal position with a cardinal value (the number of stairs climbed). All three tests were given three times at intervals of one week using sticks one time, slats another and blocks the third time (different groups were presented with the materials in different order). (Elkind, 1968, pp. 59-61.)

When Piaget used tests like these he found that the discrimination test was generally passed by four-year-olds, but he found three stages in children's ability to seriate and numerate size differences (Elkind, 1968):

> At the first stage in the development of seriation (usually at age 4), children generally are unable to seriate sticks above a small number (three or four). At the second stage (usually at age 5), children are able to make a correct seriation after considerable trial and error, but are unable to insert the second set of sticks within the completed stairway. Children at the third stage (usually at the age of 6 or 7) are able to form a stairway and to insert correctly new sticks within it. Piaget observed parallel stages in the development of numeration. Children at the first stage (usually at age 4) are unable to count correctly and cannot determine the number of stairs the doll had climbed. At the second stage (usually at age 5), children are able to tell how many stairs the doll had climbed when the stairway was intact, but not after it was destroyed. Finally, at the third stage (usually at the age of 6 or 7), children are able to say how many stairs the doll had climbed whether the stairway was together or was in pieces (p. 58).

Elkind, in replicating these tests, found significant increases in mean scores on the tests of discrimination, seriation, and numeration for groups of children of ages 4, 5, and 6. Some differences in difficulty related to different materials used (sticks, slats, or blocks) were also found (Elkind, 1968, pp. 62-67).

Wohlwill's (1968, pp. 75-104) scalogram analysis of development of the number concept was designed to provide experimental support for the theoretical premise that development of a concept proceeds from discriminative abstraction through elaboration of mental structures which leads to a state "... in which the concept exists as a purely representational symbolic entity" (p. 76).

The apparatus used in Wohlwill's study was a vertical board with three doors over which "choice cards" were hung. Behind the doors were receptacles to hold coloured chip "rewards" for correct choices. Each subject was given a "practice trial" in which the choice cards displayed two blue dots, three blue dots, and two purple concentric circles, respectively. The "sample card" also showed two purple concentric circles but larger than on the corresponding choice card. The experimenter told the subject that he would hide a chip behind one of the doors and the subject should try to find it. The subject would be able to make a correct choice every time if he looked carefully at the sample card, which would tell him the right door. If the subject made an incorrect choice he was allowed to correct it until he found the chip and he was urged to look very closely at the sample card. This correction procedure was used only in this single practice trial (Wohlwill, 1968, p. 78).

Then the subject was given a series of training tasks in which the choice cards displayed two, three, and four blue dots, respectively. The eighteen sample cards used featured two, three, or four dots in various configurations. The criterion for retaining a subject for participation in the rest of the experiment was that he should make at least six consecutive correct responses (matching sample card with appropriate choice card). If the criterion was not met in forty-eight trials, the subject was discarded from the experiment (Wohlwill, 1968, p. 80).

Successful subjects were then given a series of transfer tests which are described below in the order of difficulty hypothesized by the experimenter, beginning with the least difficult (Wohlwill, 1968, pp. 80-84).

Test A, Abstraction: The choice and sample cards used varied not only in number but also in form (little squares, circles, or triangles) and colour (green, red, blue). The sample card matched each choice card on only one of the three dimensions and the subject was forced to abstract the dimension of number from among the other, irrelevant dimensions.

Test B, Elimination of Perceptual Cues: The choice cards used were the same as those in the training series but the sample cards were rectangles divided into three or four adjacent squares.

Test C, Memory: The sample cards were those of the training series, but the choice cards were removed over the doors.

Test D, Extension: Six, seven or eight blue dots appeared on the choice cards and sample cards to give an extension of the training series to a higher portion of the number scale.

Test E, Conservation of Number: The choice cards displayed six, seven, or eight dots as in Test D, but instead of sample cards a number of small buttons were used. First, the buttons were arranged in a pattern exactly duplicating the configuration of dots on the corresponding choice card. The subject was asked to make a choice but was prevented from opening the door of his choice. Then the experimenter scrambled the buttons by hand as the subject watched. Then the subject was asked to open the door under the appropriate choice card. He was prevented from counting the buttons.

Test F, Addition and Subtraction: Instead of sample cards, buttons were again used as in Test E, but this time either one button was added or taken away before the subject was allowed to open a door.

Test G, Ordinal-Cardinal Correspondence: The choice cards used were those of the original training series, but the sample cards showed a set of eight bars arranged vertically in order of increasing length. The bars were all differently coloured and a red bar was used as a cue bar in either the second, third, or fourth position. The subject was told to look at the bars and notice where he found the red bar among all the bars. This would tell him which door was the correct choice.

The criterion for passing each of the tests was five correct responses out of six trials. Seventy-two subjects in the age range 4:00 to 7:00 completed all the tests. There were thirty-five boys and thirty-seven girls from kindergartens and primary schools in Geneva. Table I from Wohlwill (1968, p. 86) displays the number of subjects who passed each test.

TABLE I

THE NUMBER OF SUBJECTS PASSING EACH TEST

| | | |
|---|---|---|
| B. | Elimination of Perceptual Cues | 49 |
| A. | Abstraction | 46 |
| C. | Memory | 32 |
| F. | Addition and Subtraction | 24 |
| D. | Extension | 21 |
| E. | Conservation | 14 |
| G. | Ordinal-Cardinal Correspondence | 6 |

Wohlwill's scalogram analysis did suggest that the set of tests did represent a single, cumulative scale. He interpreted Tests A through C as involving matching of particular numbers with varying degrees of perceptual support. Tests E through G were considered to invoke higher-order relationships among numbers. Test D was considered to be intermediate, demanding an ability to enumerate (Wohlwill, 1968, p. 96).

The results were viewed as giving support to the postulation of three stages of development: an initial, preconceptual stage in which number is responded to in purely perceptual terms; an intermediate stage in which reduced perceptual support is needed; and, a stage in which an abstract conception of number is achieved (signified by the conservation of number and coordination between ordinal and cardinal number notions). Such a description is clearly related to and supportive of Piaget's theoretical views (Wohlwill, 1968, pp. 96-97).

A somewhat detailed description of Wolhwill's study has been given because of its original, fairly tidy approach and because it is suggestive of how standardized Piagetian tests might be developed for assessing a child's level of performance to aid teachers in planning appropriate learning activities.

Steffe (1968) found that of 132 first-grade children, the 33 that exhibited the lowest level of conservation of numerousness performed significantly less well on a test of addition problems and a test of addition facts than did the children in the higher levels.

Stommel (1967) gave 150 beginning first grade students a test based on Piaget's descriptions of the formation of number concepts and found a strong, significant positive correlation between the Piagetian predictive test scores on the SRA Greater Cleveland Mathematics Test given at the end of the year.

Brace and Nelson (1965) have reported a study involving 128 playschool children in which they found that the preschool child's ability to count is not a reliable criterion of whether or not he has achieved number conservation. They also found that concepts of cardinal and ordinal number do not develop concurrently.

Palmer (1968, pp. 6-7) produced gains in children's ability to conserve number by producing cognitive conflict in two ways: by expressing verbal surprise whenever a child responded as a conserver and by exposing nonconservers to the contradictory conclusions of their own peers who were already conservers. Two months later, the gains produced by these forms of training were found to have remained stable.

However, Piaget has some reservations about the advisability of attempting to accelerate the developmental pattern. David Elkind (1967) has relayed an interesting (secondary source) quote from Piaget that conveys these reservations very well.

> Probably the organization of operations has an optimal time... for example, we know that it takes nine to twelve months before babies develop the notion that an object is still there even when a screen is placed in front of it. Now kittens go through the same stages as children, all the same substages, but they do it in three months - so they are six months ahead of babies. Is this an advantage or isn't it? We can certainly see our answer in one sense. The kitten is not going to go much further. The child has taken longer but he is capable of going further, so it seems to me that the nine months probably were not for nothing (p. 540).

Nevertheless, many studies, particularly in the United States, have been designed around training programs for inducing conservation in nonconservers, and, hence, to accelerate the growth of logical thought. Many different approaches have been used. Some training procedures have been derived from Piagetian theory (e.g., cognitive conflict situations) and some have come from behaviourist notions (e.g., direct reinforced practice). No single training procedure appears to have been equally effective for all cases. Training effectiveness seems to be mutually dependent upon the child's developmental level, the nature of the conservation problem, and the kind of training technique used (Sigel and Hooper, 1968, pp. 258-263).

Smedslund (see Sigel and Hooper, 1968, p. 260) has found that exposure to conflict situations is effective in inducing conservation of quantity for some children, especially in situations involving conservation of discontinuous quantity (jars of beads, for example). He found similar results in regard to length conservation.

Gruen (see Sigel and Hooper, 1968, p. 261) found that a combination of verbal pre-training, involving terms such as *more* and *same*, and cognitive conflict experience to be superior to direct, reinforced practice in inducing number conservation.

Beilin (see Sigel and Hooper, 1968, pp. 261-262) found verbal instruction techniques which produce cognitive conflict to be successful in producing significant posttest differences in number and length conservation.

Sigel and Hooper (1968) believe that the training studies

> ... provide two valuable guidelines for the educator: First, they provide criteria by which curriculum units can be analyzed for sequence and relevance to the development of cognitive behaviors; and second, they assess teaching strategies relative to these curriculum considerations (p. 263).

Piaget's *The Child's Conception of Geometry* (1960) describes the child's conception of space and its measurement as developing through an invariant sequence of stages, beginning with attainment of topological concepts (perception of properties which are invariant under distortions of an object), proceeding to projective concepts (by which objects are seen as being coordinated in space rather than as isolated entities), and, finally Euclidean concepts such as angularity, parallelism, and distance (Sigel and Hooper, 1968, p. 116).

An experiment by Dodwell (1968, pp. 118-140) did, on the whole, corroborate this sequence but some difficulty was encountered in assigning any given child to a particular stage of development.

Lovell, Healey, and Rowland (1968, pp. 140-157) found a more clear-cut sequence of developmental stages than Dodwell did when they analyzed the data from twelve conservation of length, measurement, and loci experiments carried out with children of ages five to nine.

The discussion so far presented regarding Piagetian studies in elementary school settings represents only a very small sampling of the great number of such studies that have been completed. The purpose has been to convey some idea of the types of studies that have been carried out and of typical findings from these studies. Several other excellent recent studies that have not been referred to directly are included in the Additional References Section.

## Piagetian Studies in Secondary School Settings

Inhelder and Piaget, in *The Growth of Logical Thinking from Childhood to Adolescence* have described investigations that illustrate the contrasts between concrete operational thought (that typical of children of roughly seven to eleven years of age) and formal operational thought (that typical of adolescents of about age eleven or twelve and above).

In the first experiment described in The Growth of Logical Thinking (Inhelder and Piaget, 1958) the subjects were given the task of shooting a ball out of a plunger that could be aimed so that the ball would hit a target on the table after rebounding from a cushioned bank. Each subject was questioned about what he observed to see if he was able to induce from his various attempts that the angle of incidence equals the angle of reflection. Concrete-operational subjects were found to be limited to asserting specific instances of the relation and to making practical use of it to shoot accurately; they were unable to state it in its general form as a law. On the other hand, adolescents seemed to look for general principles from the outset, forming general hypotheses about the regularities they observed and putting them to experimental test (Inhelder and Piaget, 1958, pp. 3-19; Flavell, 1963, pp. 347-348).

Another experiment required each subject to explain why bodies of various densities and sizes would float or sink in water. Concrete-operational children would try to arrive at an explanation by means of a double-entry classification of observations: large-heavy, small-heavy, large-light, and small-light. They would identify the class of small-heavy objects as the non-floating class. They did not arrive at a concept of density because they could not think in terms of the volume of water displaced in order to relate the weight of the object to the weight of the water displaced. The only volumes that could be empirically observed were the volume of the object and the complete volume of water in the container. There is no empirical correlate to density for the concrete-operational child. However, formal-operational subjects, about age eleven or twelve, were able to eliminate contradictions by casting their explanations in terms of an integrated system of variables. Rather than a double-entry classification, the formal-operational subject would utilize a logical structure involving reciprocal implication and the notion of independence between two variables. He eventually would postulate that a given object floats only if its weight is lighter than that of an equal volume of water (Inhelder and Piaget, 1958, pp. 20-45; Bruner, 1959, pp. 366-367).

The adolescent's growing skill in scientific reasoning was illustrated by an experiment in which the problem was to discover the variables affecting how much a rod would bend under a given set of conditions. The materials involved and procedures employed were such that it was possible to isolate five variables as affecting the amount of bending of any particular rod: the type of metal of which the rod was made, the amount of weight supported, the length of rod, the rod's thickness, and its cross-sectional form (round, square, or rectangular). Most adolescents succeeded in differentiating the five variables. Using their combinatorial ability, they systematically tested most or all variable-present, variable-absent combinations. For example, they might vary thickness, holding the other variables constant. Younger subjects could discover some of the variables, and they did make crude attempts to test their effects, but they were unable to employ the "all-other-things-being-equal" method to demonstrate the individual effect of each variable. The disposition for systematic proof seems to be the special domain of the formal-operational thought structure (Inhelder and Piaget, 1958, pp. 46-66; Flavell, 1963, p. 348).

The experiment involving colourless liquids, some combination of which would produce a colour, has already been referred to on page 97. In this experiment, it was found that concrete-operational children did test "1 by n" and "n by n" combinations (logical multiplication), but they did not do so systematically to find out which combination(s) would produce the colour. They could not systematically eliminate variables. In contrast, formal-operational subjects generated systematic combinatorial tests to eliminate those combinations that were not adequate (Inhelder and Piaget, 1958, pp. 107-122; Bruner, 1959, pp. 367-368).

The experiments just described, and the others in The Growth of Logical Thinking, served to demonstrate that the significant difference between adolescent and pre-adolescent thinking is the ability to operate propositionally rather than concretely. The adolescent's formal thinking structures enable him to get past the errors inherent in limited concrete tests by employing the sixteen binary operations of formal logic (affirmation, negation, conjunction, disjunction, implication, etc.) to generate systematic tests by which relevant variables can be isolated (Bruner, 1959, p. 368).

K. Lovell (1961, pp. 143-153) has repeated ten of the sixteen experiments described in The Growth of Logical Thinking. Each of two hundred subjects, who were mostly between the ages of eight and eighteen, was examined individually on some selection of four of the ten experiments, with everyone doing the combinations of colourless liquids experiment described previously. Piaget's clinical approach was used and, on the basis of the protocols so collected, the performance of each student was ranked according to nine stages: one stage of pre-operational thinking, four stages of concrete thinking, and four stages of formal thinking. The results confirmed the existence of the three main stages in the growth of logical thinking as proposed by Inhelder and Piaget.

Support was found for the contention that only rarely do pre-adolescents reach the stage of formal thinking and that the ablest adolescents, though not all adolescents, do exhibit formal thinking processes. Some of the evidence suggested that the [illegible] able adolescents do not pass beyond the concrete operations stage (Lovell, 1961, pp. 143-149).

Analysis of the rankings for each student from the four experiments participated in showed considerable agreement among the levels of thinking displayed in each of the experiments. Where the kinds of ideas encountered in some of the experiments overlapped school experiences, only a minimal effect on the rankings was observed. Examination of the protocols led the investigator (Lovell, 1961) to conclude that instruction seemed to have been of greatest value when the required thinking skills were almost, or actually, available to the subject.

> If the power to think at the requisite level is not present, knowledge gained by instruction is either forgotten, or it may remain rote knowledge and be regurgitated when required (p. 51).

The experiments were found effectively to separate students classified as fast or slow learners in a wide variety of school subjects, leading the experimenters to conclude that the types of thinking processes involved in the Piagetian experiments are broadly applicable rather than only being relevant to scientific or technical problems (Lovell, pp. 149-153).

To set the stage for another Piagetian replication study, consider that Piaget has found that seventy-five percent of the subjects he has tested have evidenced conservation of substance by age seven to eight, conservation of weight by age nine to ten, and conservation of volume by age eleven to twelve. However, David Elkind's (1961, p. 551) first replication of the relevant Piagetian experiments showed that only twenty-seven percent of a group of eleven- to twelve-year-old American children had attained conservation of volume. A brief account of another of Elkind's replication studies, this time with a group of twelve- to eighteen-year-olds, is reported in succeeding paragraphs.

Elkind's second replication study was designed to determine the influences of age, sex, and IQ on the attainment of abstract conceptions of quantity (i.e., conservation of quantity or judgment of sameness despite perceptual change) in adolescents in addition to extending the replication of Piaget's experiment to the twelve- to eighteen-year-old age group. Four hundred sixty-nine Massachusetts junior and senior high school students with a mean IQ of 100.4 were given group tests of conservation. (Piaget's experiment and Elkind's first replication had used individual tests.) The materials used were two identical one and one-half inch clay balls and a small balance scale. Tests for conservation of mass, weight, and volume were given in that order. In the test for conservation of mass, it was explained that the two balls were identical in every way and that there were no tricks in the experiment. Several students were asked to verify that the two balls weighed the same by using the scale. Any students with doubts were asked to voice them, and the use of "same amount" was clarified so that all students agreed that the balls were the same. The students were then asked (Elkind, 1961):

> (a) "Do the balls both contain the same amount of clay?" (identity question); (b) "Suppose I make one of the balls into a sausage, would the two pieces of clay still contain the same amount of clay?" (prediction question); ... (after one of the balls had actually been rolled into a sausage) (c) "Do they both contain the same amount of clay now?" (judgment question); (d) "Explain your answers." (explanation question) (p. 552).

The test for conservation of weight repeated the above procedure with "weight" substituted for "amount" (after the experimenter had rolled the sausage back into a ball shape). The test for conservation of volume followed the same procedure except that "volume" and "same room or space" were used instead of "amount." A test was considered passed only if the subject answered all four questions (identity, prediction, judgment, and explanation) so that conservation was clearly evident. As a special check on the conservation of volume test, students were asked what would occur if the ball and sausage were placed in identical glasses filled equally high with water (Elkind, 1961, pp. 550-553).

As verbal misunderstandings were carefully avoided by using the experimental techniques described, it was asserted that students who failed any of the conservation tests did so solely because of inadequate conceptions. Those who failed the volume test had agreed that the balls were initially the same, but they predicted, judged, and explained that changes in shape produced changes in volume. (One cannot help wondering if this might not result from the common experience of squashing a container only to find that it no longer holds as much as it formerly did. Could there be confusion between the internal capacity of a container and the volume of substance in a solid?) Most of these same students had previously asserted that mass and weight were conserved because nothing was added or taken away, because changing the shape did not change the amount, or because what the sausage lost in width it gained in length. They failed to generalize their notions regarding mass and weight to the situation involving volume, which they treated as an entirely different problem. They failed to dissociate their subjective, sensori-motor conceptions from their objective, logico-mathematical conceptions (Elkind, 1961, pp. 553-554).

Of the students tested, eighty-seven percent demonstrated conservation of mass and weight, but only forty-seven percent had abstract conceptions of volume. In fact, only in the oldest age group (mean age: 17.7 years) did more than seventy-five percent exhibit conservation of volume. Piaget's finding that eleven to twelve-year-olds had attained conservation of volume was again not confirmed. However, a steady increase with age in the percentage of students exhibiting conservation of volume was observed (thirty-eight to seventy-nine percent for boys in groups with mean ages of 12.6 to 17.7; twenty-six to sixty-eight percent for girls in groups with mean ages of 12.6 to 17.7). At each age-level a significantly higher percentage of boys than that of girls exhibited conservation of volume. (The observed chi-square was greater than that required for significance at the 0.01 level.) A low, but positive, point biserial correlation (0.31) between IQ and success in the volume test was calculated (Elkind, 1961, pp. 554-555).

According to Piaget's theory, the majority of children of eleven or twelve should be ready to attain conservation of volume because this conceptualization requires only concrete operations, which are present in most children by age seven, and because they have had sufficient concrete experience to form abstract conceptions of mass and weight, the structural prerequisites for conservation of volume. However, the age at which the adolescent is ready to grasp conservation of volume is also the age at which formal operations are developing. Whereas concrete operations are concerned with immediate reality, formal operations are concerned with construction of systems and theories designed to investigate possibility. According to Elkind, the appearance of formal operations thus produces new interests which tend to reduce one's concern with inductive conceptualization from the physical environment in favor of more theoretical interests. The possibility of spontaneous discovery of the conservation of volume is thus reduced. The adoption of adult roles, beginning about age eleven or twelve, also leads the adolescent to be more selective in his choice of experiences. For example, the prospective scientist would likely choose different experiences from those chosen by the aspiring mechanic. It would seem reasonable to conjecture that those adolescents who attain conservation of volume despite decreased motivation have simply adopted roles conducive to the formation of such conceptions. On the other hand, though ready, many would not attain conservation of volume because their roles do not provide the necessary experience (Elkind, 1961, pp. 556-557).

The greater percentage of boys attaining conservation of volume is consistent with the hypothesis about the effects of various roles. Considering that the mean IQ of the girls in the experiment was somewhat higher than that of the boys and that there was no significant difference between the sex groups with respect to attainment of mass and weight conservation, the difference with respect to attainment of volume conservation cannot be attributed to innate differences in conceptual ability between the sexes. It seems reasonable to conjecture that the boy's traditional role of social ineptness but scientific expertise as constrasted with the girl's role of social skill and scientific aversion would give more boys and girls the opportunity to gain an abstract conception of volume (Elkind, 1961, p. 558).

Richard Skemp, a psychologist at the University of Manchester, has developed a theory of mathematics learning that is derived from a Piagetian point of view. In the course of testing his theory, which is particularly relevant to mathematics learning in the secondary school, he designed an experiment to measure student ability to form and to manipulate concepts and operations. He emphasized manipulation in the design of the study because conscious manipulation of concepts and operations requires that these be formulated, a reflective process, whereas the formation of concepts and operations may or may not require reflective activity. Furthermore, he considered that the chief ability required in mathematics at the secondary school level is the skillful combination and use of known concepts and operations. Accordingly, Skemp (1958, pp. 156-157) developed two two-part tests to measure student ability to form and reflectively manipulate concepts and operations.

For the first test involving concepts, fifteen properties were chosen that could be possessed or not by simple line drawings (e.g., being curved, continuous, closed, dotted, or self-crossing). For Part I of this test, Skemp drew three exemplars labelled "Examples," three non-exemplars labelled "Not Examples," and three "Test Figures" for each of the fifteen properties. The subjects were asked to try to discover the property held in common by the "Examples" and to indicate which of the "Test Figures" possessed this property. The criterion was the ability of the subject to use the concept, thus indicating he had formed the particular class-concept, rather than his ability to verbalize or formulate it. Since Part II of the first test was to be a measure of the student's ability to manipulate the concepts of the first part, a preliminary trial with twelve-year-olds was carried out, and only those concepts which most children were able to grasp were retained (Skemp, 1958, p. 157).

Part II of the concepts test was designed to measure the student's ability to manipulate concepts. For this part, thirty-five pairs of the properties used in Part I were chosen, and three exemplars of each double property were drawn. The three non-exemplars drawn for each double property had, respectively, only one of the properties, the other property, and neither property. The subjects were asked to decide whether or not each of three "Test Figures" possessed both properties. To grasp and demonstrate each double concept the subject would have to think reflectively, not only being aware of the single class-concepts but deliberately combining and separating them (Skemp, 1958, p. 158).

The operations formation part, i.e. Part I, of the second test consisted of an answer sheet and a demonstration sheet giving three simple abstract-line-figure examples of each of fifteen operations such as clockwise rotation through a right angle, reflection in a horizontal line, and interchanging the numbers of figures in two groups. The subjects were asked to discover what each operation was from the demonstration sheet after some similar operations had been explained. They were then asked to carry out the operation on three specified figures on the answer sheet, drawing the results in provided blank spaces (Skemp, 1958, p. 158).

Part II of the operations test, manipulation of operations, involved combining and reversing the operations from Part I. Both of these processes involve reflective activity and are relevant to mathematics, in which practically every operation has its reverse (or inverse) and in which successful problem solving depends on suitable choices from among the combinations of one's available operations. Of the fifteen problems given, the first five involved carrying out two operations in combination, one after the other, on three figures and showing the results. The second group of five problems involved carrying out the reverse of a single operation on three figures, and the third group of five involved conceiving of the reverses of two operations and then carrying the reverses out in combination one after the other. Before beginning Part II, the subjects were given the answers to Part I, after their own answers had been collected, and these were explained to ensure that the basic operations were understood. Demonstration sheets were used by the students in Part II, as in Part I, so that no memorizing was required (Skemp, 1958, p. 159).

Fifty-fifth form (twelve- to sixteen-year-old) students wrote both parts of the two tests and they all wrote the same general certificate of education (G.C.E.) mathematics exams. The correlations between the students' G.C.E. mathematics scores and their scores on Part II of the first test and Parts I and II of the second test were 0.58, 0.42, and 0.72, respectively. The multiple correlation between the G.C.E. mathematics criterion and an optimum weighting of scores from Part II of both tests was 0.77 (Skemp, 1958, pp. 162, 223). Especially considering the nature of Skemp's tests, the observed correlations with the mathematics achievement test are quite remarkable and support the notion that the exercise of reflective intelligence is important in mathematics.

## Educational Implications

A number of interesting implications from Piaget's cognitive theory of intellectual development can be found in an article by Irving Adler (1966, pp. 581-584). The present paragraph summarizes some of those implications. The mathematical experiences a child is given at any age should be experiences he is ready for in terms of the stage of mental growth he has reached, and they should help prepare him to advance to the next stage. Before introducing a child to a new concept, one should test him to see if he has the prerequisites for forming the concept, and, if he does not, he should be provided with appropriate developmental experiences.* Especially in the lower grades, concepts should be built from appropriate concrete experiences, rather than by using the easier, but less effective device of "telling." To help a child overcome his errors in thinking, provide him with experiences that will expose the errors, thus assisting the process of accommodation that will eventually lead him to cope adequately with the situation at hand. Flexible thinking is based on reversible operations. By analogy, it would seem beneficial to teach operations in inverse pairs and to stress their relationship (e.g., the relationship between addition and subtraction). Children in the stage of concrete operations can be helped to gain a better grasp of relations among subsets of a set if they are given experiences in manipulating sets of objects to explore relationships among sets, subsets, intersection, union, and hierarchical inclusion. Combinatorial analysis is based on the formation of Cartesian products of sets. Children can be readily taught systematic ways of forming these products by using tree diagrams and rectangular arrays. Since mental growth is encouraged if one is given opportunities to see things from many points of view, teaching should give children the opportunity to use a wide variety of approaches in tackling problems. For example, in teaching geometry, not only the traditional synthetic approach should be used, but approaches using analytic techniques, vectors, and isometries of the plane could also be included. As mental growth is associated with discovery of invariants, a systematic search for the features of a situation that remain unchanged under a group of transformations should aid in developing awareness of and understanding of the relationships involved in the situation. Since the onset of formal thinking occurs at about age eleven or twelve, it would appear psychologically sound to introduce short units of deductive reasoning from hypotheses as early as the sixth grade.

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*The Educational Testing Service is carrying out a very interesting project in New York City in which ingeniously devised tests and developmental experiences with a Piagetian flavour are being used with first graders in place of the usual I.Q. and "readiness" tests.

An interesting and insightful further observation made by Adler is that the concrete operations used by an individual are "concrete" in the sense that they are mental operations involving some system of objects and relations that is perceived as real by the person. What is "concrete" is relative to the person's past experience and mental maturity. While the kindergarten child considers the union of two beads with three beads as a concrete operation but the addition of $2$ and $3$ as not, the introductory algebra student considers $2 + 3$ as concrete but not $x + y$. The student of introductory abstract algebra considers the additive group of integers as concrete but does not consider the concept of an abstract group to be concrete. So the progression goes, and it is evident that "concrete" operations are used not only in the concrete operations stage, in which they are the most advanced operations of which the child is capable, but also at all succeeding levels of learning. In the development of new concepts at any level it is necessary to proceed from what the learner perceives as concrete to what to him is abstract (Adler, 1966, p. 584).

Piaget has stated that an individual's apparent failure to grasp the most basic concepts of elementary mathematics stems not from a lack of any special aptitude but rather from affective, emotional blocking or inadequate preparation. Furthermore, the frequent failure of formal education can be traced to the fact that it begins with language, illustrations, and narrated action rather than real, practical action. Preparation for mathematics education should begin in the home with the encouragement of concrete manipulations that foster awareness of basic logical, numerical, and mensurational relationships. This practical activity should be systematically developed and amplified throughout the primary grades until it takes the form of elementary physical and mechanical experiments by the time secondary education begins (Piaget, 1951, pp. 95-98). In a similar vein, Eleanor Duckworth (1964) has said:

> You cannot further understanding in a child simply by talking to him. Good pedogogy must involve presenting the child with situations in which he himself experiments, in the broadest sense of

> that term -- trying things out to see what happens, manipulating things, manipulating symbols, posing questions, and seeking his own answers, reconciling what he finds at one time with what he finds at another, comparing his findings with those of other children (p. 497).

In much the same line of thought, Piaget has been cited as having taken issue with those who recommend that children be taught the "structure" of a subject area so that they will be able to relate individual aspects to the general structure. During a discussion period at a conference, he said (see Duckworth, 1964):

> The question comes up whether to teach the structure or to present the child with situations where he is active and creates the structures himself ... The goal in education is not to increase the amount of knowledge, but to create the possibilities for a child to invent and discover himself ... Teaching means creating situations where structures can be discovered; it does not mean transmitting structures which may be assimilated at nothing other than a verbal level (p. 498).

In another context, Piaget has warned against the danger often inherent in school learning of leading the child to false accommodation to words or to authority rather than to reality as it presents itself. It is preferable for a teacher not to correct a child's schemata, but to provide situations that will lead the child to correct them himself (Duckworth, 1964, p. 498).

In discussing teacher education, Piaget has been reported as saying that even adults can learn better by doing than by being told about such things as how to teach effectively. Furthermore, he is of the opinion that prospective teachers should have the opportunity to question children in a one-to-one situation so that they will realize how difficult it is to make oneself understood. They should also have occasion to pursue an original investigation to determine what children actually think about some problem. In endeavouring to communicate individually with a number of children in this way, the prospective teacher may be able to overcome the illusion that he can talk successfully to a whole class of children at once (Duckworth, 1964, p. 498).

In line with the view that intellectual development brings a gradual transformation of overt actions into mental operations, a key concept in Piaget's theory, a teacher would do well to assist the internalization and schematization process by having students perform actions with less and less direct support from external entities. For example, the child might be led to operate directly on physical objects, then on pictorial representations, then on cognitive anticipations of operations not actually being performed, and so on, until the original external operations take place internally and independently of the

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environment. Furthermore, since social interaction is essential in developing the multiperspective view essential for rationality and objectivity, group activities in the form of projects and discussions should be encouraged (Flavell, 1963, pp. 368-369). When Piaget says that the child should be actively engaged in the learning process he means not only should the child actively manipulate and experiment with materials but he should also actively compare his findings with those of other children. Arguing with his peers forces the child to reason with himself (Sigel and Hooper, 1968, p. 431).

As Millie Almy (1966) has observed, Piaget's theory makes a strong case against allowing a child to learn a procedure for getting an answer without being able to retrace his steps or without being able to think of alternative approaches to arrive at the same result. To do so "... is to encourage the erection of a verbal superstructure that may crumble under even minimal cognitive stress" (p. 132). Almy (1966) also provided an interesting example from a first-grade teacher whose students

> ... have been successfully completing exercises that required them to supply the sums for rows and columns in a series of diagrams. Then comes a set of exercises in which the sums are presented and they must write in the appropriate figures for the rows and columns. The numbers involved are small and the context provided by the diagrams has not changed. Nevertheless, the children who presumably have been relying largely on memory in the previous problems are thoroughly confused. The teacher commented that these children are not really "operational" in their thinking. Piaget's analysis helped her to understand the problem as it was viewed by the children. It also led her to question whether these children had had sufficient concrete experience to build a stable concept of number, or whether the earlier exercises had been insufficiently varied (p. 132).

Piaget's method of interrogation, in which a suspect answer from a child is carefully probed by the use of other questions and the original question is repeated, rephrased, or related to manipulation of relevant materials, can be used by teachers to help a child to reveal his own thoughts rather than parrot a response that is thought to be the one the teacher wants (Almy, 1966, p. 133).

It is important to be aware that a child is often influenced more by his own way of looking at materials and objects under consideration than he is by the questions asked of him about the objects. An example related by Almy (1966) of a child turning an adult's question into his own way of thinking had to do with a group of metal blue cars and red cars. When the child was asked if there were more metal cars or more blue cars, he replied that there were more blue. When he was asked to repeat the question asked he said: "Are there more blue cars or more red cars?" (pp. 133-134).

Millie Almy (1966, p. 134) has found that even though teachers of young children often find Piaget's ideas difficult at first, they invariably sense the relevance of such ideas to their teaching and, if they work through one experiment with one child, they are not inclined to stop there. Their extended experimentation leads to better understanding of Piagetian ideas and, more importantly, to greater insight into the nature of children's learning tasks.

## Sample Mathematics Teaching Approaches

An approach to the teaching of mathematics that ties in very well with Piagetian ideas is that developed by Robert Davis (1965) of the Madison Project. The Madison Project materials stress: learning of really fundamental mathematical ideas such as variable, function, graph, matrix, isomorphism, and so on; an active role for the student; learning of concepts and terms in context (beginning with tasks rather than definitions); enabling students to search for mathematical patterns and "discover" them for themselves; a nonauthoritarian teacher role; intrinsic motivation; leading students to want to go beyond what happens in class and to feel that mathematics is "fun" and "exciting" (Davis, 1967, pp. 3-4). Davis feels very strongly that children must develop their own mathematical systems:

> If a child has discovered concepts himself, has devised techniques himself, and has elaborated a mathematical system himself, he really knows "how" and "why" it works in a profound way that is not possible when the system is handed to him or told to him... Children really learn only by some kind of active participation. The best math students have always actively developed mathematical ideas in their own heads -- as David Page says, one of the objectives of the "new math" programs is to get every child to think about mathematics the way the best students always have (p. 1).

At the risk of oversimplifying the approach, in what follows an attempt is made to briefly illustrate how Davis leads students to generate "The Axioms of Arithmetic and Algebra." He has successfully used the approach with ninth grade students as well as fourth and fifth graders and at various grade levels in between. One of the key ideas (Davis, 1964) in the approach is "... active, creative, original student participation... The students choose sets of axioms, and the teacher argues with them about limitations of their chosen set. The teacher accepts "wrong" answers and waits for some students to challenge them" (p. 9).

After having experienced "informal exploratory experiences" and game situations in which are developed concepts of "open sentence," "the meaning of 'equal'," "truth sets," "use of variables," and "principle of names" (any true statement remains true if any name in the statement is replaced by another name for the same thing), the students are led into generating an extremely long list of identities of their own invention.

Typically, to begin with, the students are asked (Davis, 1967): "Can you make up an open sentence that will become true for every legal substitution?" (p. 171). Students often initially make up open sentences like:

$$x \times 0 = 0$$
$$x \times 1 = x$$
$$x + x = x + x$$
$$x + x = x \times 2$$

After a number of such open sentences have been contributed by the students and discussed, the term identity is introduced to refer to this special kind of open sentence and the students are encouraged to make up and record long lists of identities. They find out how to generate more complex looking identities from simple ones and the class is divided into two teams which take turns competing for points by making up identities and having the other team try to decide whether the open sentences are or are not identities. A cumulative list of all the identities that come up is kept.

Lists of identities that are all or nearly all specific instances of more general identities are presented (as the work of students) and eventually some student sees the general pattern and suggest a "super-identity" that handles all the particular cases. For example, the identities

$$3 \times x = x \times 3$$
$$4 \times x = x \times 4$$
$$5 \times x = x \times 5$$

are generalized to

$$y \times x = x \times y.$$

Notions of implication are introduced in the context of reducing a number of given statements without losing any information. For example, asked to shorten this list (Davis, 1964):

(a) My cousin plays in the Little League.
(b) Only boys play in the Little League.
(c) My cousin is a boy (p. 166).

some students will point out that if you say the first two statements, you don't need the third, i.e. no information is lost.

Eventually, the students are confronted with the following problem (Davis, 1967):

> Take your list of identities and shorten it as much as possible, without really losing anything. What does your final list look like (p. 180)?

They use generalization and implication in the process of shortening their list and eventually, after much discussion, arrive at a list of "axioms," which come very close to being the set of field axioms, from which the students feel they can derive all possible identities. Here is a sample derivation (Davis, 1964):

Theorem: $A + (B \times C) = (C \times B) + A$

Proof: $A + (B \times C) = A + (B \times C)$

$A + (B \times C) = (B \times C) + A$

$A + (B \times C) = (C \times B) + A$

Q.E.D. (p. 167).

What is truly remarkable about the process is that the students develop their own mathematical system and that they do so with great enthusiasm. An interesting problem to them is whether or not they can write a derivation from their basic axioms for a statement like $(x + y)(x + y) = x^2 + 2xy + y^2$.

Another example of a teaching approach that fits in very well with Piagetian ideas can be found in a recent article by Sigurdson and Johnson (1968), "A Discovery Unit on Quadratics." The article should be of particular interest to classroom teachers of mathematics because it details eleven activities which have been used successfully in leading several groups of Alberta grade eleven students to discover all of the important properties of quadratics virtually on their own. Each activity began with a broad question which the students were asked to seek answers for. Given a chance to explore the problem individually for a while, the students were asked to hypothesize solutions and evaluate each other's hypotheses. The teacher accepted each hypothesis without evaluative comment and acted merely as a discussion leader. Once the students had focussed on the valid hypotheses from among all those considered, they were led to participate in "summing up" and "practice" sessions.

The first activity consisted of using an overhead projector to display a coordinate grid and graph of the parabola generated by $y = x^2$. The students were asked to come up with a rule relating the coordinates of each of the points in the parabola. Among the hypotheses offered were: a verbal description of the symmetry of the curve; $x^2 - y = 0$; $y = x^2$; and a verbal description of the "over by one's and up by 1, 3, 5, 7, ..." feature.

The second activity centered on investigating the effects on the rule of moving the parabola horizontally or vertically. The students were given

$y = x^2$ parabolas plotted on onionskin paper along with background grids so that they could move the parabola around to suggest hypotheses. Moving the parabola "n" units up the vertical axis was seen to change the rule to $y = x^2 + n$ (generated from hypotheses about motion of 1, 2, 3, ... units). Attempts to describe the effects of horizontal motions eventually led to a $y = (x - n)^2$ hypothesis, but not until several incorrect hypotheses were struggled with and tested.

Next, the student-initiated problem: "How do you make a graph wider or narrower?" was investigated, using graphs of $y = x^2$, $y = \frac{1}{2}x^2$, and $y = 2x^2$ to generate ideas. Moving these parabolas both vertically and horizontally at the same time led very naturally to expressing the rules relating the coordinates in "vertex form" (i.e., $y = (x - \Delta x)^2 + \Delta y$).

A fourth activity involving the graphing of quadratic functions expressed in "standard form" ($y = ax^2 + bx + c$) led to student desire to "complete the square" to get the rule in the vertex form which they preferred.

Seven other activities described in the article focussed on "axis of symmetry," "range," "x and y intercepts," "roots of quadratic equations," "quadratic formula for roots," and "maxima and minima problems."

Yet another approach to mathematics teaching and learning that is consonant with a Piagetian point of view and that is presently generating considerable interest is that used in "mathematics labs." Kieren and Vance (1968) have written of their experiences in using a mathematics lab approach with students in grades seven and eight. Their article contains suggestions for suitable kinds of activities and lab organization.

An excellent source of ideas for Piaget-based activity oriented teaching approaches is the set of "Nuffield Mathematics Teaching Project" Teacher's Guides (see Nuffield Mathematics Teaching Project).

Perhaps the common feature found in each of the examples of mathematics teaching approaches cited is that each in some way manages to make classroom learning more like real life learning and, hence, more effective, more lasting, more useful, and more enjoyable.

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# COMPUTERS IN MATHEMATICS INSTRUCTION[1]

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The use of computer technology within instructional systems is relatively new. A decade ago almost no one was seriously engaged in the type of journal writing which today is commanding serious attention. None of the more than 60 funded projects of today were in existence and most were probably not even in the early planning stages ten years ago. In fact, even a modest survey of the literature reveals that almost all of today's activity has a history dating post-1964. With such a narrow temporal base you might be inclined at this point to assume a very skeptical posture. To be sure, our experiences have been that real educational change has an entropy-like character (a sort of "lag of the seasons" not unlike Minnesota in April--a full month after the official start of spring!). You may be counting on this inertia to protect you from change for awhile. Or we in mathematics education may feel that we have been a part of a "revolution" already and now is not the right time for another...a sort of "let us catch our breath for awhile" attitude. Or you may see the onset of computers into the classroom as another ivory-tower fad which its advocates herald as the "messiah of education" but which you feel will eventually enjoy the same mediocre impact of so many previous innovations.

In my opinion, there can be little doubt that computer involvement in instruction, in one form or another, will make a substantial impact on education during the next decade. Further, computers can, if used correctly, bring about profound and far-reaching improvements within education---and even with learning itself. These are strong statements which most assuredly require some further clarification. I will attempt to provide some information in support of these claims by considering three specific areas. First, how can computers assist with instruction and why is there a rapidly growing interest in it today? Second, what is the current state of the art in terms of the research and development activity, particularly in mathematics education? And third, what are some of the trends and future prospects for using computers in mathematics instruction?

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## Use of Computers to Assist in Instruction

What is computer-assisted instruction? Some confusion and disagreement with this label has already occurred. In most circles "CAI" is the acronym used to describe the learning situation where a student sits at a terminal connected electronically to a computer. Stored in the computer is the learning program which is the complete package of information, instructions, and logic with which the student will interact during his learning session. The terminal which serves as the interface between the computer and the student is typically a typewriter keyboard and either a paper roll or a TV-like screen upon which the communications to and from the computer are recorded. Some terminals also involve various audio and/or video projectors, such as video or tape recorders, and slide, film, or movie projectors. Terminals with TV screens, often called CRT's for "cathode-ray tube," may also accept input from an electronic light pen which the student may use to point to his response.

Partially because a great deal of the current activity is focusing on the pedagogical approach advocated by Skinner, Crowder, and other programmed learning enthusiasts, the term "CAI" is often used to connote only using the computer to administer the kind of frames found in a good programmed learning text. It is true that a great deal of the work with computers has been initiated by people well versed in programmed learning methodology. As a result of their influence, there may be a temptation to say that CAI is merely the next logical extension of programmed instruction and basically provides just a new hardware capability. I believe that this would be an extremely limited view to be taking at this time.

The literature describes the computer giving assistance in the instructional process in a wide variety of ways. Suppes has identified three levels of pedagogical involvement in the approach most often labeled CAI (Suppes, 1966; Suppes, 1968b; Suppes, Jerman, and Groen, 1966; Suppes, Jerman, and Brian, 1968). At the most superficial level of student-computer interaction are drill and practice systems which he describes as merely supplements to a regular curriculum taught by a teacher. In the next level of interaction, called tutorial systems, the aim would be to significantly assume the responsibility for teaching. The third and deepest level of interaction seen by Suppes would allow a genuine dialogue between student and the learning program. He notes that such dialogue systems exist only as very elementary prototypes. It is these three types of systems which have generally become known as CAI today. In each case, considerable programming by an author or teacher has been performed. Today, one can find only a few drill and practice and tutorial systems which are being used to administer mathematics instruction. Some of these systems will be reviewed in a later section.

The most widespread approach to using the computer in mathematics instruction is often called computer-assisted problem solving. This use has been a natural outgrowth of the way the computer was first used by the scientific community: that is, to accept, process, and print output to a program which makes use of the machine's capability to store and access large data bases, make quantitative decisions, and perform detailed computations, all at very great speeds. Several mathematics educators have suggested that the activity of designing computer algorithms offers a pedagogical approach that can improve the learning of mathematical concepts. Some of the speculation and research dealing with computer-assisted problem solving will be reviewed in a later section also.

The two approaches I have described thus far (CAI and problem solving) picture the computer being accessed by the student. A different use views the computer as a tool in making instructional management decisions. This could mean that students never interact with the actual computer during instruction. Rather, the machine may be programmed to administer tests, record student responses, and perform the kinds of data-crunching necessary to provide the teacher with student diagnostic information to be used in evaluating learning and prescribing further instruction. Of course, such instructional management processes are typically an integral part of CAI systems which administer stored instructional material.

To be sure, these three categories described as CAI, problem solving, and instructional management do not neatly exhaust the kinds of uses to which computers are now being put. For example, a project of the Board of Cooperative Educational Services of Erie County, Buffalo, New York, (Eisele and Harnack, 1967; Harnack, 1967) developed a number of computer based resource units designed to provide the teacher with assistance in offering more individualized instruction to his students. Each unit is a computer printout of suggestions to the teacher geared to (1) the specific instructional objectives chosen by the teacher for the total class, (2) the specific objectives for each pupil, and (3) the individual characteristics of each pupil. Once the computer is fed specific instructional objectives and the characteristics of an individual pupil, the machine generates a resource guide printed as a content outline, suggestions of large group activities, small group activities, instructional materials, and measuring devices for each objective chosen for that pupil, and suggestions for individual instructional activities and materials. I suppose this use of computers as information storage and retrieval devices, contributing to instructional decision making, might be viewed as part of the instructional management approach.

In a conventional physics course at Florida State University students were invited to use a special CAI review service which consisted mainly of test items from previously used physics examinations. How does one categorize such a use? And then there is the appealing area of simulation and gaming which will surely continue to receive attention. Simulation is already used in the aerospace industry, conventional aircraft pilot training, driver training, electronic manikins for medical training, and in physics and chemistry laboratories. Computer-controlled games are usually simulations involving situations of competition or conflict. Among the more interesting are the economics games developed for elementary students in Westchester County, New York, (Wing, 1964) and the Inter-Nation Simulation developed at the PLATO project of the University of Illinois (see Hickey, 1968).

## Current Interest in Computers for Instruction

Why is there a reasonably widespread interest is using computers to assist in the instructional process? Robert Bundy (1967) believes that CAI is really the result of a number of converging technologies, including programmed learning, audiovisual communications, the data processing field, and data communications. Atkinson and Wilson (1968), from the Stanford CAI Center, attribute the rapid rate of growth of CAI to "the rich and intriguing potential of computer-assisted instruction for answering today's most pressing need in education--the individualization of instruction" [p. 73]. Other factors cited by them are the development of programmed instruction, the mushrooming of electronic data processing in general but in particular the advent of time-sharing systems, and the increasing aid to education by the federal government. Sarnoff, president of RCA, observes that the technology revolution in education is being hastened by the staggering growth in the volume of knowledge and the enormous increase in the number of students. He claims that by 1970 every third college graduate will have to become a teacher if the present pupil-teacher ratio is maintained.

Stolurow (1962,1968), who has a long history of involvement with CAI, having been in the pioneer center at the University of Illinois and currently director of the center at Harvard, sees in CAI three key capabilities: (1) Individualizing instruction, (2) doing research on teaching under controlled conditions with the ability to collect detailed records of student performance, and (3) developing ways of assisting authors in the development of instructional materials. Other applications seen by Stolurow, aside from instruction, include the development of teaching models, curriculum planning, man-machine relations, and evaluation of student performance.

Another recognized prophet is Gerard (1965,1967), Dean of Graduate Studies, University of California at Irvine. He lists these benefits CAI will bring to the student: (1) better and faster learning since the student can time his learning at his convenience, go at his own pace, and catch up missed time; (2) better teaching at many levels and in many areas; (3) personalized tutoring; (4) automatic measurement of progress; (5) and the opportunity to work with vastly richer materials and more sophisticated problems. For the teacher, the system (1) takes away a great deal of the drudgery and repetition; (2) allows him to be updated effectively; (3) encourages frequent changes in the actual material used; and (4) makes more time available for teacher-student contact.

Several writers recognize the usefulness of CAI as a research tool in education. Bundy (1967) submits that "the concept of a learning laboratory is the single most important reason for the developing interest in CAI" [p. 345]. This emphasis probably stems from a recognition that much of what can now be said about CAI is couched in the future tense--what CAI might eventually be able to do as an instructional device--but the immediate job at hand with CAI is to increase our understanding and control of the variables of learning. The computer is particularly well suited to this task. Consider, for example, the capability of the computer for storing and manipulating data. As a student works through a CAI program, a complete trace of his learning path is recorded. This data tells the researcher exactly what the student did in responding at every step and gives the researcher an exact measure of the response latency at each step. The data can be presented to the researcher during the learning session, if desired, and thus can be used in a formative evaluation to modify the presentation while it is in progress. Further, the data can be quickly manipulated in many interesting ways to reveal various types of statistical summaries and comparisons for diagnosing the learning variables under study, such as comparisons with prior performance, effectiveness of diagnostic questions in the learning sequence, and correlations between step size and efficiency of learning.

The computer's capability as a communications control device adds still another dimension to its usefulness. Many forms of educational media can be employed in a single learning program, and their presentation, either separately or simultaneously, can be operated under the complete control of the computer. This can provide a richness and versatility in learning that is difficult to achieve today, certainly in the conventional classroom. But most important, it allows us to study the effects of a near endless variety of interactions of media presentation modes and subject matter, and consequently, to learn better how such modes should be differentially used.

The capability of the computer, coupled with data communications technology, immensely aids the editing and distribution problems related to developing and circulating educational programs. Many students in widely scattered locations can use the same computer at the same time on a time-sharing basis. Much data is thus simultaneously being acquired which can greatly facilitate the editing function, even while the program is in session if desirable. Changes that must be made are easily stored in the central computer, and thus the distribution problem is considerably simplified.

Stolurow (1964), succinctly summarizes many of these points as follows

> ...the automation of instruction by means of a computer-based system provides education with a genuine laboratory facility. Education no longer needs to settle for makeshift laboratories as it has been forced to use up to this time. This contribution of CAI is a substantial one. Armed with effective tools in the form of an educational laboratory, instructional materials and methods can be used repeatedly so that studies can be replicated and variations specified and introduced with control...Furthermore, all responses which the student makes can be recorded so that the learning process itself comes under controlled experimental observation and becomes available for experimental and conceptual analysis. In addition, the responses which the students make are recorded in a form that can be immediately processed by the same or a different computer. Consequently, control, convenience, speed, and accuracy are all achieved at levels not heretofore possible for instructional research. The advantage of the computer for research on instruction is clearly in the data collection and data reduction capabilities. Computer-based systems can add enormously both to the rate at which research on instruction can be accomplished and to the accuracy of the results produced (see Bundy, 1967, p. 346).

What about the approach called computer-assisted problem solving? Why is there such a rapidly growing interest in this instructional approach? The National Council of Teachers of Mathematics has maintained an attentive ear to the possibilities of using computers in mathematics instruction for several years. The text Computer-Oriented Mathematics, An Introduction for Teachers (1963a) and the Report of the Conference on Computer-Oriented Mathematics and the Secondary School (1963b) are indicators of the Council's early concern. The productive and active Computer-Oriented Mathematics Committee developed important guidelines published in the article "Computers for School Mathematics" (1965) and in the pamphlet Computer Facilities for Mathematics Instruction (1967). The recent pamphlet An Introduction to an Algorithmic Language (BASIC) (1968) is an indicator of the emphasis on the use of a simple programming language and on the design of several levels of algorithmic sophistication. The March, 1969, Arithmetic Teacher was devoted to CAI and the new department on "Computer-Oriented Mathematics" began in the April, 1969, Mathematics Teacher. In addition to this activity of NCTM, one can find analogous activities being conducted by the Committee on the Undergraduate Program in Mathematics of the Mathematics Association of America (1964), by the National Science Teachers Association (Darnowski, 1964), and by the School Mathematics Study Group (1966) and the Minnemast curriculum projects (Rosenbloom, 1963). The CRICISAM Project at Florida State University is developing a computer oriented calculus curriculum (Center for Research, 1968).

Several speculative statements of the effects of using this style of instruction can be identified. For example, John Kemeny (1966,1967,1968) developer of the language called BASIC and mathematician identified with the well-known Dartmouth time-sharing computer center, submits that students use the computer to more effectively learn those procedures taught theoretically in class. In the publication Needed Research in Mathematics Education, Kemeny (1966) states

> I feel that the right attitude is to teach them the algorithms in principle and then the right way to do the algorithm in practice is to program it for a computer. Thus the computer is being used in such a way as to force the student to explain the given algorithm to a computer. If a student succeeds in this, he will have a depth of understanding of the problem which will be much greater than anything he has previously experienced [p.10].

Bruce Meserve (1968) notes that the "availability of time-sharing procedures is making classroom use of computational facilities both educationally and financially feasible" [p.113]. Regarding the effect on learning, he conjectures the following: "Students who acquire a working introduction to algorithmic languages while in high school gain an opportunity for greater insight into both their high school and their college mathematics. Most high school students strive to learn patterns and general cases. In this sense, the use of the computer may be considered as a 'next phase' in the student's mathematical development" [p.113].

Hoffman (1963), from Wayne State's computer center, commenting on the use of mathematical settings to teach computer programming, recognizes the possible gains in learning the mathematics involved:

> In the experiences which I have had along this line, it has been quite clear that students acquire astonishingly high insight into the mathematical problems which have programmed for solution on a digital computer. It is easy to see why this should be so, as a computer program must take into account all possible cases of a general nature....To generalize this, by using the computer the student is forced to acquire insight into the general algorithm, the conditions under which it applies, and the general class of problems to which a given procedure can be applied as well as to those special cases to which the general solution is not applicable. This is clearly what we have always been trying to teach mathematics students, and contact with the computer makes it easier to accomplish the desired ends [p.20].

These viewpoints suggest the utility of a computer program as a dynamic problem solving tool. The use of a natural programming language and the organizational features of a program provide the students with a setting for imposing precision on themselves. Any computer program, good or bad, is an active object. With it, the student can command the computer to do something which he can observe, study, and modify. Thus, the activity of programming should foster an experimental approach toward solving mathematical problems.

Dorn (1967), from the University of Denver, advocates the use of computers to provide such a laboratory setting in mathematics. In a beautiful article which focuses on the Fibonacci sequence and continued fractions, Dorn (1968) notes that

> ...the computer can be introduced in a traditional classroom environment and does not require any major change in teaching practices....the mathematics computer laboratory can supplement the usual mathematics lecture and recitation much as a physics or chemistry laboratory supplements lectures in those subjects...
>
> It appears quite feasible then, for many schools to establish a mathematics laboratory to carry out experiments that motivate the student to study certain mathematical topics and to help him develop his mathematical intuition. It is important to keep in mind, however, that such a laboratory is best used to teach mathematical concepts and to extend the range of the mathematical topics that can be taught. The laboratory is not intended to be a device to teach programming or computer science, although some knowledge of those subjects will be a by-product of the laboratory's use. To quote from Computers in Higher Education, a report of the President's Science Advisory Committee:'...it is important that computers be used to extend rather than displace the students grasp of other subject matter.' [p.79].

One additional point of view seems particularly worth noting when speculating on the problem-solving usage of computers in school mathematics. It seems to me that pervading the use of computer algorithm design in the teaching of mathematics is the recognition that formal knowledge is only part of what we try to impart to our mathematics students. Heuristic knowledge, concerned with the art of solving problems, is particularly emphasized when students write programs to solve problems. The design of computer algorithms seems ideal for experiencing such heuristic precepts as "formulate a plan," "find a related problem," "observe special cases," or "simplify the conditions." Thus, writing computer programs seems a natural context to make more concrete the approach to teaching usually attributed to George Polya. This approach to problem solving is described by Feurzig and Papert (1960) in a paper devoted to the prospects of using programming languages as conceptual frameworks for teaching mathematics. They state:

> Solving a mathematical problem is a process of construction. The activity of programming a computer is uniquely well-suited to transmitting this idea. The image we would like to convey could, roughly speaking, be described thus: A solution to a problem is to be built according to a preconceived, but modifiable, *plan*, out of *parts* which might also be used in building other solutions to the same or other problems. A partial, or incorrect, solution is a useful object; it can be extended or fixed, and then incorporated into a larger structure. These remarks are true of mathematical thinking in general. But in must contexts they are too subtle to be meaningfully taught. An important example of how programming brings them down to earth is the use of the process of *debugging* programs as a paradigm for the crucial--but neglected--aspect of mathematical thinking that has to do with turning errors to positive advantage [p.12].

I will briefly cite some of the background information dealing with instructional management. Why are numerous educators and projects directing attention to the use of computers in designing and implementing instructional management systems? What is the nature of the speculation being cited for such uses of computers? Stated simply, the efforts to organize individualized instruction, particularly continuous progress programs (of the type used in the Duluth elementary schools), has led to a recognition of the inadequacy of teachers to monitor and prescribe such instruction. Cooley and Glaser (1968), from the University of Pittsburgh's Research and Development Center which has produced the Individually Prescribed Instruction (IPI) curriculum, describe instructional decision making this way:

> All teaching involves decisions about how instruction should proceed. Particularly characteristic of individualized instruction is the necessity for instructional decisions relevant to each student. The differential decision-making function in individualized instruction is a central issue. These decisions require a great variety of information about the individual student, such as (1) what criteria of competence should be applied? (These

> have traditionally been stored in terms of test grades, teacher judgments of quality, etc.) (2) what is the background of the student? (This has been stored in the student's written record in terms of intelligence test and aptitude test scores.) (3) how does a student proceed in his learning? (This is usually the teacher's impression of the student as slow or fast, or attentive or distractable, and rarely takes the form of documented information.) (4) what instructional means are available for teaching certain lessons? (This has been catalogued in the teacher's head or on a resources list.) [p.5].

In the model of individualized instruction envisioned by Cooley and Glaser, a sizable amount of information is needed on a daily basis for each student. It is obvious that some form of assistance is necessary to help the teacher collect, store, and act in terms of such data. They outline six features of a computer-management of instruction (CMI) model:

(1) Specification of goals, subgoals, and decision nodes.
(2) Measurement and diagnosis of the initial state or behavior with which the student enters an instructional situation.
(3) The assignment of instructional alternatives.
(4) Continuous monitoring and assessment.
(5) Adaption and optimization.
(6) Evolutionary operation.

The details of such a CMI model include knowledges which do not seem clearly available given today's theories of instruction. But I will discuss this point later.

## WHAT IS THE CURRENT STATE OF THE ART?

Let us now turn to reviewing the state of the art. What is the experimental evidence to support or refute these uses of computers in instruction? How far have the developments progressed? In this review I will examine basically information which deals with mathematics instruction; of course, some information may have implications for teaching mathematics but which has not been determined using school mathematics as a setting.

In the October 1968 document Computer-Assisted Instruction: A Survey of the Literature (Hickey and Newton, 1968) available from ENTELEK (an information processing center originally funded by the Office of Naval Research) there are listed and described 1[illegible] university centers, five industrially-based centers, 15 military projects, and eight public school and consortia dealing with CAI research, development, and application. Approximately 500 individual public and private schools have at least a limited CAI capability and over 50 separate organizations have fully operational time-sharing systems.

Charp (1967) reported the results of a 1965 survey of 65 schools using computer programming in their curricula at a conference on The Computer in American Education (Bushnell and Allen, 1967). In the same conference, Karl Zinn (1967a) another pioneer of CAI activity at the University of Michigan, provides a very comprehensive annotated listing of the multitude of projects involved in the programmed learning type CAI. Snater (1968) reported the responses of 50 school systems selected because of their involvement in using the problem-solving approach. Thus, it would appear that considerable activity is underway in both categories of CAI and problem-solving.

What measures of effectiveness of the computer-assisted problem-solving approach are available? The Computer Utilization Program of the Altoona Area School District, Pennsylvania, is probably the only high school project in the country to operate its own full-scale time-sharing teleprocessing system owned by the district and located in the high school (Information Service Department, 1969). Started in 1964 as a small, government-funded instructional unit for some 200 vocational students, the General Electric system has been expanded in well planned steps until today the computer is self-supporting, serving 4000 students in 16 different schools across five central Pennsylvania counties.

Computer utilization in the mathematics and science department is integrated into the curriculum. BASIC is taught to all eighth and ninth grade students and FORTRAN is taught to all mathematics and science students in the senior high school. It is felt that if instruction in programming is introduced early in junior high school, the programming work can be incorporated directly into the curriculum. The student learns just enough programming technique each year to program typical problems given in the particular mathematics and science classes. In solving their problems, students do all their work outside of class. Students are required to make an analysis of their problem, draw a flow chart, and write a documented program in one of the languages. The students program is punched onto paper tape at one of the numerous teletypewriters and transmitted to the computer. Teachers frequently asked, "Where are we going to find the time to cover all the required material and still teach programming?" Through actual experience with a test group, however, it was found that less time had to be spent on drill work to reinforce a difficult concept since the student had to analyze his problem completely before programming. More material could be covered since less time was required for the tedious calculations. Also, problems that were bypassed before because of their complexity and lengthiness were now tackled with success. The one bit of experimental evidence reported from the Altoona project states that experimental groups who have been taught BASIC and computer usage for algebra problem solving have attained a higher level of learning, expecially in arithmetic reasoning, than control groups not similarly exposed.

Jesse Richardson, Massachusetts State Dept. of Education, describes a similar time-sharing project which uses the TELCOMP language in grades 6, 9, and 11 (see Hickey and Newton, 1968). At the time of this paper, a final report is not available.

The only available comprehensive research results which I was able to locate have stemmed from the two year Computer-assisted Mathematics Project, better known as CAMP, conducted at the University of Minnesota High School. Operating under a grant from the General Electric Foundation, David Johnson (1966) directed a development and research program at grades 7, 9, and 11. The purpose of the project was to identify appropriate material in the existing mathematics program which might be more effectively studied by designing computer programs. The grant supported the s[illegible] teletypewriter terminal and the time-sharing contracts over the two years.

Kieren (1968) reported a two year study conducted with eleventh grade mathematics students in the CAMP project. The 36 students from the first year and the 45 students involved in the second year were randomly assigned to either a computer or a noncomputer group. For purposes of analysis, the classes were blocked into average and high previous achievement groups. The two classes were taught in both years by the experimenter. The textbook used in the course was the SMSG Intermediate Mathematics. The difference in the treatments was that the computer class learned much of their mathematics by writing BASIC programs which involved the problems, concepts, and skills from the regular mathematics course while the noncomputer group did not use the computer in any way. Various measures of mathematical achievement were obtained during each of the two experimental years. Methods of analysis of variance and covariance were applied to these measures in order to test hypotheses of no differences in group means after instruction. Also, the proportions of correct responses were examined for 348 test items used in the second year.

Kieren reported one rejection out of eight of the null hypotheses of no treatment effects during the first year. This rejection favored the mean of the computer class on the standardized Contemporary Mathematics Test, Advanced Level. No significant differences of treatment by previous achievement level interaction were found for the eight significance tests.

During the second year, the achievement of the computer class was found to be significantly higher according to the means for the Unit Test on Quadratic Functions when the analysis involved the pretreatment STEP 2B and Unit Test on Functions scores as covariates. Using the same eleven tests and the analysis of variance procedure, Kieren rejected the null hypothesis of no differences due to treatments for the means of the Unit Test on Trigonometry and the COOP Trigonometry Test in favor of the regular class. As in the first year, the test of interaction revealed no significant differences. However, an inspection of the cell means suggest that the computer seemed to be relatively more effective for students of average previous achievement. Kieren (1968) states: "It is of interest to note that the rejection favoring the computer class came in a unit which was deemed 'extensive' in terms of computer use. The rejections in favor of the regular class come in the trigonometry unit where the computer use was deemed 'moderate' in the description of the treatment" [p.125].

The null hypothesis of no difference in the proportion of students correctly responding to a test item was rejected for 43 of the 348 items included. In studying these items, it appeared that the computer had little positive effect on simple skills such as computation with complex numbers and geometrical treatments of trigonometry. On the other hand, the computer "seems to make its strongest contributions in the areas of complex skills, organization of data and drawing conclusions therefrom, and the study of infinite processes" [p.127-8]. Kieren suggests that additional research needs to be done involving more subjects and using carefully detailed written materials to implement this use of the computer.

I was able to complete a second study (Hatfield, 1969) as a part of the CAMP project. My research involved seventh graders over a two year period. The design and instructional procedures are analogous to those described for Kieren's study. Subjects were randomly assigned to treatments. In the computer treatment, students wrote computer programs involving the same mathematical content taught in the non-computer treatment. Special supplementary materials were written and used with the computer group to teach BASIC programming, to identify the content to be programmed, and to guide the writing of programs, the study of output and the processes of "debugging" or refinement toward a more general algorithm. Comparisons involved several constructed tests, commercial standardized tests, and a selected problem solving test. Revisions of the supplementary materials and constructed tests were sufficiently extensive to require that each year be treated as a separate experiment. For purposes of analysis, each treatment was blocked into three levels of previous achievement using scores on the *STEP Mathematics 3A* test. Analysis of variance was used to compare main effects due to treatments and differential effects of treatments across previous achievement levels. Comparisons of proportions of students responding correctly to a test item were used to explore in greater detail the particular contributions of each treatment. Each of these items had been classified as a "skill," "concept," or "problem" item by three mathematics educators who applied a construct specification developed by the experimenter.

During Year 1, the effect due to treatment as measured by group means was significant for only one (*Numeration Systems*) of the eleven criterion tests. This difference favored the noncomputer treatment with the greatest difference in cell means occurring at the low previous achievement level. During this initial unit, the computer students also learned the BASIC programming procedures which seemed to interfere with concurrent study of numeration systems. The equality of item proportions was tested for 266 items. The proportions of correct responses in the control group was significantly greater on 22 items, while on 19 items the computer group was favored. On six of the eight significant "skill" items, the control group was favored.

During Year 2, the means analysis of treatment effect revealed significance on one (*Elementary Number Theory*) of the six unit tests and two (*Contemporary Mathematics Test* and *Thought Problems*) of the six posttreatment tests. These significant differences all favored the computer treatment. Comparisons of cell means on these three tests revealed that the high and average previous achievement computer groups were especially favored. The number theory unit was recognized as a particularly relevant setting for the use of the computer. The emphasis of this unit was on exploration and inquiry with problems involving many laborious calculations. The orientation was to use the computer as a laboratory tool to explore a number of interesting number theory settings. The proportions for 327 items revealed that the computer group scored significantly better on 25 items while 13 items favored the control group. The computer group was significantly favored on 12 of the 16 problem items and 10 of the 16 concept items.

The results from these two studies do not support computer-assisted problem solving as the optimal approach to be taken in all settings. At the same time, there is evidence that these seventh and eleventh grade students could learn to program the computer to study their school mathematics. Furthermore, in several particular settings, the computer-approach did result in significantly improved performance.

While the heaviest overall usage of computers in instruction has been in the category commonly called CAI there seems to be experimental evidence dealing with mathematics instruction from only a few of the centers. One of the most carefully planned and extensively documented projects has been the drill-and-practice CAI project for arithmetic from the Stanford center. Now in their fifth year of work, Pat Suppes, Max Jerman, and others have compiled an impressive performance record. In addition to a wealth of articles and center reports (Suppes, 1966; Suppes, 1968a; Suppes, 1968b; Suppes, Jerman and Groen, 1966; Suppes, Loftus, and Jerman, 1969), Suppes, Jerman and Brian (1968) have authored a detailed textbook account of the 1965-66 arithmetic program. To give some idea of the extent of Stanford's current project, the February 1969 issue of their CAI Newsletter (Jerman, et al., 1969) reports that nearly 250 terminals are now located in schools and homes in California, Mississippi, Kentucky, and Washington, D.C. In the period from October 9 through December 20, a total of 83,007 tests and lessons were given with 57,399 administered to the users in Mississippi alone!

A comprehensive review of the detailed results in the evaluations being produced by this project will not be attempted here. An evaluation of the 12 Mississippi schools of 1967-68 using the Stanford Achievement Test as a pretest and posttest revealed that at all Grades 1-6 the comparison of mean gains in grade placement scores significantly favored the group which had received the brief daily drill administered by the computer. The difference is largest in Grade 1, where in only three months the average increase in grade placement was 1.14 for experimental students compared to .26 for control students. The effect was the least for the fourth and fifth graders. It is noteworthy that this approach to drill and practice has been implemented by the New York City public schools this year where over 6000 students are reported to be receiving brief arithmetic and reading drills daily using an RCA computer system.

I cannot leave an account of the Stanford CAI center without according some acclaim to the formulation of the theoretical linear structural models for predicting response and latency performances in the CAI arithmetic drills (Suppes, 1967a; Suppes, 1967b; Suppes, Hyman and Jerman, 1967). It is exactly such contributions which will be necessary before educators can ever hope to implement a system of computer-managed instruction. Of course, should the computer never become a tool in instructional management, the knowledge of the type generated by Suppes and his staff should have relevance in the design of instruction, whatever the medium.

In the 1967-68 Report from the CAI Laboratory of the University of Texas, Gibb (1968) reported the results of three studies involving elementary mathematics students indicating some significant work in computer-assisted arithmetic instruction in the Texas center.

Let me draw your attention to at least one project which seems to have implications for future uses of computers in assisting with instructional management. Hively, Paterson, and Page (1968) report one of the first non-trivial applications of what Osburn (1968) has named "universe-defined achievement testing." In this approach, the basic arithmetic program used in federal Job Corps centers was analyzed into precisely defined domains of behavior. Item forms, consisting of general forms together with a specification of generation rules, precisely defined the set of all test items which were taken to represent a diagnostic category. The studies reported were tests of whether the defined item forms represented distinct, homogeneous classes of behavior which in turn could provide the foundation for detailed diagnosis and remediation. Although a computer was not used to generate the particular tests from the various item-form universes, the authors note that a computer could be easily programmed to do this. It is easy to see the utility of such a testing system for evaluation in a learning system aimed at providing differential placement.

## Some Next Steps

As must always be the case, I have only touched upon the variety of current activity involving computers in mathematics instruction. I would propose the following as conclusions, trends, and unanswered questions for teachers and researchers.

(1) Mathematics instruction, particularly at the secondary and college level, will continue to be the setting for the investigation of the approach called computer-assisted problem solving. Future investigations should seek answers to questions like the following: Is this way of studying mathematics effective for general mathematics and applied mathematics students? What are

the transfer effects of using this computer approach to data organization, decision making, and corrective refinement of decisions in other disciplines, such as the physical and social sciences? What are the relationships between the style and organization of algorithm designs and the learning styles of students? Are there difficulty levels associated with particular computer algorithm structures? What are the attitudes toward the learning of mathematics and toward this particular characterization of man-machine systems? Are students indeed more motivated to learn mathematics while using this computer approach? Are there particular features of a programming language that make it the best selection for different students? What are the advantages and effects of using an instructional mix of CAI-tutorial material with problem-solving and algorithm design? Ennis (1962-5) of Cornell University has submitted that several aspects of Piaget's formulation of the development of logical thought are inadequate. He has posited a different model. Can the processes of computer algorithm design be described as characterizations of such models of reasoning and logic as Piaget and Ennis have discussed?

(2) CAI at the drill-and-practice and tutorial levels will continue in the main to be an activity of heavily funded Research and Development centers. The use of CAI as an instructional laboratory for researching questions aimed at more adequately describing and predicting the learning of mathematics is assured. Widespread usage of CAI instruction will depend in most part upon how rapidly hardware people can bring down the inflated costs of terminals and communication. One possiblity of a major breakthrough is currently claimed by Bitzer (1968) of the University of Illinois CAI center. He and others have developed a plasma display tube capable of color and rear screen projection from microfiche which can be driven over coaxial cable at great distances, which will be low in production cost, and which can support related media devices. A description of this device and Bitzer's claims are part of the proceedings of the September, 1968, Conference on Computer-assisted Instruction sponsored by the NCTM, NSF, IBM and the Pennsylvania State University (see Heimer, 1969). Much of the developmental work in CAI in mathematics will focus on special "excursion" material, dealing primarily with remediation and enrichment. Should such material become generally available, teachers will be faced with the challenging problem of determining how such available instruction is to be properly blended into more conventional classroom instruction. Assistance in this problem may come in part from the work just beginning in the third area of computer use I have described: namely, instructional decision making.

(3) Computer-management of instruction (CMI), described in its current theoretical state, holds probably the most significant impact for educational environments. The utility of employing computers in management of instruction is contingent upon the continued surge toward designing more individually sensitive learning opportunities. I believe that the current activity aimed at individualization is not simply a fad, but rather is the first insurgence of what will be a major educational revolution. The feeble predictive knowledge we now have about how children learn mathematics must be vastly extended before we will be able to design what will approximate optimal instruction for each individual.

In closing, let me propose that the community of mathematics educator-researchers formulate a plan for developing a Center for the Study of Mathematics Learning (see Suppes, 1967a). I believe that such a think-tank could effectively focus the diverse activity and accelerate the present trickle of basic, formative research. Such a Center, staffed with a reasonable number of the insightful people currently interested in studying mathematics learning, could be the capability required to overcome the information gap that prevents us from having any real assurance that what we do in teaching a particular child is what is best for him. The advent of the computer seems to hold the promise of overcoming many of the manpower and economic barriers in individualized tutorial instruction. The real question is whether we will be able to intelligently use the capabilities which computer systems will offer to us.

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# Problem-Solving and Creative Behavior in Mathematics[1]

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Problem solving and the related, more elusive subject of creative behavior have received substantial attention from mathematics educators during the past five years. The role of problems in developing students' mathematical activity was chosen by the International Commission on Mathematical Instruction as one of three topics for discussion at the 1966 International Congress of Mathematicians in Moscow. Reports to the Commission by the Conference Board of the Mathematical Sciences (1966) in the United States and by the Association of Teachers of Mathematics (1966) in England highlighted the importance of problems in mathematics instruction and indicated that we need to know much more about using problems to stimulate independent and creative thinking.

The Cambridge Conference on School Mathematics (Educational Services Incorporated, 1963) urged curriculum developers to devote more time and energy to the creation of problem sequences, with special emphasis on problems that can be used to introduce new mathematical ideas. The coming "second round" of curriculum revision anticipated by the Cambridge Conference will clearly require careful specification of criteria for these problem sequences (Baughman, 1967). As complex, challenging mathematical problems assume a more central role in the curriculum, more research that makes use of such problems will be needed.

Calls for more research are easily made. A more difficult task is to locate, amid the vast, amorphous literature on problem solving and creativity, studies that might inform future research. As the literature reviewed here suggests, the topics of problem solving and creative behavior are not being investigated systematically by mathematics educators. Fortunately, some psychologists have used mathematical tasks, or mathematics-related tasks, in their research on higher cognitive processes. Though the use of mathematical problem material in psychological studies appears at times to be almost fortuitous, these studies fill some of the gaps in current research in mathematics education and may provide some of the direction that is now lacking.

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[2] James W. Wilson, University of Georgia, served as consultant on the preparation of this review. Richard Pocock assisted in the initial search of the literature.

## Bibliographies and Reviews

Problem solving in elementary school mathematics was the subject of several recent reviews. Riedesel (1969) listed 83 outstanding articles and research reports from the past fifty years and noted some obvious implications of this research for arithmetic teachers. Gorman (1967), in a more systematic and critical analysis, identified 293 studies on word problems conducted between 1925 and 1965. Only 37 of these studies, mostly doctoral theses, were deemed "acceptable" according to Gorman's criteria, which emphasized the control of factors affecting internal validity. Generalizations drawn from the acceptable studies were as frequently contradicted as confirmed by results from the remaining studies. A monumental survey by Suydam (1967) of published research on elementary school mathematics from 1900 to 1965 yielded 84 studies of problem solving, the largest single category in her topic classification scheme. Suydam, too, found conflicting results and a generally low quality of design and reporting.

Similar surveys at the secondary school level would be useful, if only to document the dearth of published studies on problem solving at the higher grades. Comprehensive general reviews of problem-solving theory and research are given by G. Davis (1966) and in the volume edited by Kleinmuntz (1966).

Investigations of creative behavior in mathematics have not been collected and reviewed apart from other creativity studies. The bibliography by Razik (1965) and the review by Arasteh (1968) are useful general references on creativity research. Davis, Manske, and Train (1967) have summarized the literature on methods for training creative thinking.

## Problem-Solving Ability

Without necessarily conceiving of the ability to solve mathematical problems as a unitary phenomenon, one can learn something of its nature by examining the relationships between an individual's success in problem solving and other characteristics of his thinking and personality. Studies of problem-solving ability range from straightforward comparisons of group performance to intricate factor analyses.

## Individual Differences

Tate and Stanier (1964) analyzed the performance of good and poor problem solvers on tests of critical thinking and practical judgment. Subjects were 234 junior high school students whose scores on a composite measure of problem-solving ability, including mathematical and quantitative reasoning problems, deviated markedly from a regression line of problem solving on IQ. On the critical thinking tests, the poor problem solvers tended to avoid the judgment "not enough facts" and to make unqualified "true" or "false" judgments. On the practical judgment test, they tended to select answers having a high affective component. Tate and Stanier argued that the errors may stem from response sets having a temperamental rather than an intellectual basis. Of special note, too, are the striking differences between good and poor problem solvers on a test in which they had to identify the missing data in arithmetic problems (Tate, Stanier, and Harootunian, 1959).

Sex differences in problem-solving ability were studied in a novel way by Sheehan (1968), who used a multivariate analysis of covariance with a step-down procedure to examine the relative influence of concomitant variables on differences in achievement. After five weeks of instruction in algebra, 107 high school freshman were given a criterion test divided into subtests designed to measure higher- and lower-level cognitive processes. Superior performance by the girls on the lower-process test disappeared as sequential adjustments were made for their initial superiority in eighth-grade mathematics achievement, algebra aptitude, and pre-instructional knowledge of algebra. Further, the boys showed superior performance on the higher-process test after the adjustments were made. The results indicated that sex differences in ability to learn complex problem-solving skills may be masked if criterion tests emphasize the acquisition of information.

Koopman (1964) found that although boys and girls at each of the grades from nine to twelve were about equally successful in solving arithmetic problems, the girls at each grade were less confident of their solutions than the boys. The twelfth graders were better able to judge their solutions correctly than were the ninth graders. Robinson (1964) found no sex differences or differences between seventh and ninth graders in the ability to justify a mathematical generalization by means of a proof, but she did find differences attributable to the tasks and to whether or not the student was in a general mathematics program.

## Related Skills and Abilities

Success in solving word problems in mathematics clearly depends upon skills in reading and computation, but the relative contribution of these skills is not so clear. Balow (1964), using a factorial design with sixth graders at four levels of reading ability and four levels of computational ability found that both factors were associated with problem-solving ability as measured by the Arithmetic Reasoning subtest of the Stanford Achievement Test. There was no significant interaction between the two factors. When IQ was taken as a covariate, similar results were obtained, except that computational ability showed a stronger effect than reading ability. Martin (1963) found that each of the factors of reading comprehension, computation, abstract verbal reasoning, and arithmetic concepts was correlated with problem solving as measured by the Arithmetic Problem-Solving Test of the Iowa Tests of Basic Skills given to fourth and eighth graders. The partial correlation between reading and problem solving with computation held constant was higher at both grade levels than the partial correlation between computation and problem solving with reading held constant. As Martin suggests, the relationship between problem-solving ability and its underlying skills, particularly higher-order verbal skills, is probably more complex than had been supposed.

Two factor analytic studies of problem solving in mathematics were synthesized by Weredelin (1966), who rotated the two factor matrices to a congruent structure. The loadings on the five factors isolated in each study were virtually identical. Tests of problem solving loaded most strongly on a General Reasoning factor and to a lesser extent on a Deductive Reasoning and a Numerical factor. The other factors, Space and Verbal Comprehension, were unrelated to problem solving. Werdelin's subjects were high school boys, so his analyses do not permit an examination of sex differences, but Very (1967) has demonstrated with college students that the mathematical abilities of males are more differentiated and more easily identified than those of females. Similar patterns were also found at grades nine and eleven, with a general increase in number of factors with age (Dye and Very, 1968).

## Affective Variables

An experimental study of test anxiety by Jonsson (1965) illustrated the importance of affective factors in problem solving. Identical mathematical problems from the Sequential Tests of Educational Progress were embedded in two different test versions, one containing easy problems and one containing difficult problems. Classification variables in addition to test version were scores from the Test Anxiety Scale for Children. An analysis of covariance was done separately for each sex, with arithmetic reasoning as the covariate. The sample consisted of 358 sixth graders. Results showed some interaction of test anxiety and test difficulty, especially for girls, to the detriment of the performance of highly-anxious subjects taking the more difficult version. The results could be used to bolster arguments for matching achievement tests more closely to estimated performance but also suggested that anxiety may act generally to distort findings in studies of complex problem solving.

Evidence on the influence of motivational factors was reported by Gangler (1967), who found that college students who were informed that their work on a series of learning tasks in symbolic logic would count toward their mathematics course grade performed less well on learning and problem-solving tasks than students who were not so informed. The effect was greater for students of high intelligence than for students of low intelligence.

Kellmer Pringle and McKenzie (1965) argued, on the basis of a pilot investigation, that a less competitive school environment may reduce frustration and stress among low-ability pupils. They found less problem-solving rigidity among low-ability children in a child-centered progressive school in England than among comparable children in a traditional school. These findings are suggestive at best; the joint influence of affective factors and intelligence on mathematical problem solving warrants further study.

Additional References: Kennedy and Walsh (1965); Lindgren et al. (1964).

## Problem-Solving Tasks

When problem material varies all the way from geometric proofs to matchstick puzzles, one wonders whether consistent generalizations can be made about problem solving in mathematics. Some reassurance that laboratory tasks have relevance to classroom work has been given by Olson et al. (1968), who obtained significant correlations between grades in seventh-grade mathematics and performance in paired-associate learning, discrimination learning, concept of probability, conservation, verbal memory, and anagram tasks. The results were complicated by sex differences, especially when IQ was partialed out, but a clear and unexpected finding was that the more complex, school-like tasks, such as conservation of volume and anagram solving, had no greater predictive validity than the simple rote-learning tasks. Variations in problem material do make some difference, however, and several investigations were aimed at assessing their effects.

## Problem Content

A perennial issue in mathematics education concerns the use of problems that are closely related to students' interests and experience. Travers (1967) asked 240 male high school freshmen to choose and solve one of two problems that were identical in structure (numbers used, operations required, etc.) but different in setting. The subjects showed strong preferences for "social-economic" situations (e.g., selling hot dogs) compared with "mechanical-scientific" situations (e.g., testing spark plugs) and "abstract" situations (e.g., solving secret codes). The last situations were particularly unpopular. General mathematics students showed stronger differential preferences than algebra students, and there were some tendencies, although slight, for problem preferences to be related to vocational interests as expressed on the Kuder Preference Record.

The hypothesis that disadvantaged children would perform relatively better on problems whose content dealt with lower needs, such as food and shelter, than on problems whose content dealt with higher needs, such as mastery and education, was tested by Scott and Lighthall (1967). Need content of the problems was not related to degree of disadvantage of third and fourth graders. A principal components analysis of the data suggested that factors associated with the difficulty and the mathematical content of the items, rather than the need content, accounted for differences in performance.

## Problem Structure

Steffe (1967) investigated the effects of two variations of the language used in a problem on its difficulty. Twenty one-step addition problems were presented orally to ninety first graders in individual interviews. In ten of the problems the names for the two sets to be combined and the total set were the same ("There are four cookies on one plate and two cookies on another plate. How many cookies are on the plates?"), and in ten of the problems the names for the three sets were different ("Mary has four kittens and two goldfish. How many pets does Mary have?"). Half of the subjects were given problems in which an existential quantifier was used at the beginning of the problem ("There are some cookies on two plates"), and half were given problems without the quantifier. The presence of the quantifier had no effect on problem difficulty, but the problems having a common name for the sets proved to be significantly easier than the problems having different names for the sets. Steffe concluded that curriculum developers should give more attention to problem situations in which the sets are described in different words.

In a study of three factors hypothesized to affect problem difficulty, West and Loree (1968) found that decreasing a problem's redundancy (reducing the repetition of data) and decreasing its selectivity (adding irrelevant data) made the problem significantly more difficult for seventh and ninth graders, and that decreasing a problem's contiguity (putting the data some distance apart) made it significantly more difficult for seventh, but not for ninth graders. Placing the question at the beginning rather than at the end of the problem statement did not improve performance significantly, according to Williams and McCreight (1965), who asked fifth and sixth graders to solve problems of both types. Burns and Yonally (1964), using two- and three-step problems, found that fifth graders were more successful when the data were presented in the order used to solve the problems than when the data were presented in another order. Subjects low in arithmetic reasoning ability were more affected by the reordering of data than were subjects with high ability.

Suppes, Loftus, and Jerman (1969) studied the relative contribution to problem difficulty of six variables: (1) *operations*, the minimum number of different mathematical operations needed for a solution; (2) *steps*, the minimum number of steps (applications of operations); *length*, the problem length, in words; (4) *sequential*, whether or not the problem could be solved by the same operations as the preceding one; (5) *verbal-clue*, whether or not the problem contained a verbal clue to the operations needed; and (6) *conversion*, whether or not conversion of units was necessary. Data were obtained from 27 bright fifth graders, each of whom solved 68 word problems presented in a computer-assisted instructional program. The six variables accounted for 45 percent of the variance in performance on the problems, with the sequential, conversion, and operations variables making the greatest contribution, in that order. The sizable contribution of the sequential variable suggests that the context in which a problem is embedded may be almost as influential as the problem's structure in determining its difficulty.

The wording of problems is a structural feature usually assumed to be related to their difficulty. Thompson (1967) wrote ten arithmetic problems at two levels of readability as defined by the Dale-Chall and Spache readability formulas. Mean reading grade levels for the two sets of problems were 2.7 and 8.7. Subjects were 368 sixth graders, half with IQ's above 110 and half with IQ's below 100 (as measured by the *California Test of Mental Maturity*). Thompson reported that the effects of readability and mental ability on performance were interactive. Although readability affected performance at both levels of mental ability, it had a greater effect with subjects of low mental ability.

Stull (1964) investigated the effects of reading problems aloud to subjects. Five classes at each grade from fourth to sixth received auditory assistance while taking an arithmetic test; five classes at each grade did not. The test comprised four subtests that measured: (1) knowledge of quantitative relationships found in social situations, (2) ability to recognize missing data, (3) ability to disregard irrelevant data, and (4) ability to make appropriate assumptions from the data. Only on the subtest of ability to disregard irrelevant data did the subjects who received auditory assistance perform better than the others, and this effect was significant for the girls but not for the boys. On the other subtests, differences were not significant. The results suggest that problem difficulty is more a function of reasoning ability than of reading skill.

*Additional References*: Early (1967); James (1967).

### Problem-Solving Processes

Since the solution of a problem--a mathematics problem in particular--is typically a poor index of the processes used to arrive at that solution, problem-solving processes must be studied by getting subjects to generate observable sequences of behavior. Psychologists have devised numerous techniques for studying problem solving (Bourne and Battig, 1966), but mathematical problems are seldom used in such research. Nonetheless, much of the recent literature on problem-solving processes has relevance for mathematics educators.

## Developmental Changes

Piaget's theories on the growth of logical thinking have served as both focus and touchstone for developmental studies. Sharples et al. (1968) examined the relationship between second and fourth graders' ability to solve transitivity of length problems and their ability to solve logic problems embodied in switch-light tasks. A scalogram analysis indicated that transitivity of length was prerequisite to the form of seriation used in solving the logic problems. The fourth graders, however, failed to show a hypothesized superiority on the transitivity task. Informal interviews suggested that the fourth graders' performance may have been hampered because they were searching for a "trick"--an observation reminiscent of Weir's (1964) finding that seven- to ten-year-olds often perform at lower levels on problem-solving tasks than younger and older children. The seven- to ten-year-old may be "at a point in development where his ability to generate complex hypotheses and employ complex search strategies is growing at a faster pace than his information-processing ability, which catches up only at a later age" (Weir, 1964; p. 481). Impressive evidence has begun to accumulate that young children can learn and transfer various complicated problem-solving strategies (Stern, 1967; Stern and Keislar, 1967; Wittrock, 1967).

Freyberg (1966) used an objective test designed to measure the development of Piagetian concepts in a two-year longitudinal study. The subjects were 64 boys and 87 girls, six to nine years of age. Scores on the concept test were as predictive of arithmetic computation and arithmetic problem-solving ability two years later as was Primary Mental Abilities Test mental age. Furthermore, a regression analysis showed that the concept test added significantly to the prediction of arithmetic attainment by mental age.

O'Brien and Shapiro (1968) demonstrated that the ability to recognize logically necessary conclusions is not the same as the ability to test the logical necessity of a conclusion. First, second, and third graders were given one of two tests: a 100-item test, devised and used previously by Hill, in which subjects responded "yes" or "no" to questions on premises from sentential logic, classical syllogisms, and the logic of quantification; and a 100-item modification of the first test in which subjects responded "yes," "no," or "not enough clues" to 67 unaltered items and 33 items altered so that the third response was correct. The results on the first test confirmed Hill's previous findings of growth from ages six to eight and a generally high level of performance at these ages in ability to recognize logical necessity. The results of the second test showed, except in the case of the altered items, no significant growth with age in the ability to test logical necessity, and a significantly lower level of performance overall. Subjects in all three grades tended to avoid selection of the "not enough clues" option when it was correct.

## Strategies of Inquiry

The now-classic A Study of Thinking (Bruner, Goodnow, and Austin, 1956) has stimulated mathematics educators (e.g., Rosskopf, 1968) to ask whether the strategies employed in concept attainment tasks apply to problem solving in mathematics. Concept attainment studies, however, have seldom used mathematics problems, perhaps because such problems do not fit the information-gathering model very well. In solving a mathematics problem, one often has most of the information he needs; the question is what to do with it. Research on strategies of inquiry can tell only part of the story, but it may provide some leads for identifying more general problem-solving strategies.

Rimoldi et al. (1964) developed and validated a variety of criteria for assessing performance on information-gathering tasks. They found that scores derived from the logical structure of the problem were more useful and discriminated better than did scores based on group norms. Practice on information-gathering tasks improved subjects' performance, and male college students performed better by most criteria than did male high school students. A subsequent study (Rimoldi, Aghi, and Burger, 1968) showed similar increases in performance with age from 7 to 13 years. Neimark and Lewis (1967) have suggested that increases in logical information-gathering strategies with age are attributable not to differential learning rates but to increases in the number of subjects at each age who are capable of using a strategy. In other words, the acquisition of logical strategies is essentially all-or-none for each individual--a hypothesis worthy of further test.

Recent work by Dienes and Jeeves (1965) has given what may be a productive new focus to the search for strategies. Instead of studying subjects' behavior as they gain information about concepts, Dienes and Jeeves studies how subjects reorganize stimulus material into structures. Using a task in which the card a subject played, together with the card shown in the window of an apparatus, determined which card would appear next in the window (a binary operation), Dienes and Jeeves presented embodiments of mathematical groups (the two group, the cyclic group of order four, and the four group) as games. The subject played a card and predicted which card would appear next in the window until his predictions were consistently correct. When asked how the game worked, subjects gave three types of evaluations: operational, indicating that they regarded the card played as operating on the card in the window; pattern, indicating that they divided the game into parts in which similar combinations of cards yielded similar outcomes; and memory, indicating that they had merely memorized each combination.

Dienes and Jeeves identified a strategy of card choice corresponding to each type of evaluation and found that a subject's use of strategies was reflected in his report. The hierarchy of evaluations (operator--pattern--memory) was validated in various performance measures. Adults tended to use more strategies and to give fewer memory evaluations than did children. Subsequent work (Jeeves, 1968), using groups with 3, 4, 5, 6, 7, and 9 elements, confirmed the findings of the earlier study that children were better able to particularize (from larger group to smaller group) than to generalize (from smaller group to larger group) and that children had more difficulty generalizing than did adults. In Dienes and Jeeves' view, S-R models are inadequate for explaining subjects' performance on these tasks, and some kind of structural learning must be postulated.

### Heuristic Methods

Modern interest in heuristic--the study of the methods and rules of discovery and invention--is due principally to Polya (1957, 1962, 1965), who has set forth maxims for problem solving which, he postulates, correspond to mental actions. Evidence for the validity of Polya's observations on the problem-solving process has come most strikingly from work on computer simulation of human behavior. Programers have found that the incorporation of general heuristic rules, such as working backward or using a diagram, not only makes problem solving more economical, but it also results in performance by the computer that closely resembles the behavior of human subjects struggling with similar problems.

Information-processing approaches to the study of problem solving (see Hunt, 1968; Newell and Simon, 1965) have frequently used mathematics problems. Paige and Simon (1966) compared the protocols of subjects asked to think aloud as they solved algebra word problems with the processes used in a computer program for translating English sentences into equations and then solving them. Analysis of the protocols showed that subjects used some kind of internal representation of the physical situation described by the problem in framing their equations. When given "contradictory" problems in which equations can be written even though the problem is physically impossible, subjects consistently differed in their ability to detect the contradiction. Paige and Simon concluded that good problem solvers are more likely than poor problem solvers to discover contradictions of this sort, an observation in conflict with Krutetskii's (1969) finding that for some contradictory problems, capable secondary school students and adults make more mistakes than less capable subjects, who use a concrete interpretation and thereby discover the fallacy. The difference in results may be attributable to the Soviet educational practice of teaching mathematics problems as representative of certain "types." Krutetskii's more capable subjects apparently saw the contradictory problems as embodying a type, recalled the type solution, and then mechanically substituted the (illogical) data into the solution. Kennedy, Eliot, and Krulee (in press) gave a five-step description of the act of solving algebra word problems: (1) assimilate the problem statement, (2) assess the information given, (3) identify the relationships among elements to form equations, (4) incorporate unstated logical or physical assumptions, and (5) solve the equations. Twenty-eight high school juniors, equally divided between the sexes and between honors and regular mathematics classes, were given three numerical and three word problems and asked to think aloud as they worked. The honor students did not differ from the average students in their ability to assess the information given or to identify the relationships among elements, but the honor students were better able to make inferences about unstated logical and or physical assumptions, in confirmation of the Paige and Simon observation mentioned above. An additional finding by Kennedy, Eliot, and Krulee, that the less able students had a greater tendency to formulate problems in the order the data appeared, supported Burns and Yonally's (1964) report that students low in arithmetic reasoning ability had greater difficulty with problems when the data were reordered.

Kilpatrick (1967), using a system based on heuristic processes identified by Polya, analyzed problem-solving protocols of 56 junior high school students in relation to their performance on a battery of aptitude, achievement, and attitude scales. Systematic styles of approach to spatial and numerical problems, similar to styles identified factor analytically by French (1965), proved to be unrelated to processes used in solving word problems. Subjects who attempted to set up equations (they had not yet had an algebra course) were significantly superior to the others on measures of quantitative ability, mathematics achievement, word fluency, general reasoning, logical reasoning, and a reflective conceptual tempo. Subjects who did not use equations varied in their use of trial and error. Those subjects who used the most trial and error were higher than the others in quantitative ability and mathematics achievement. Those subjects who used the least trial and error had the most trouble with the word problems, spent the least time on them, and got the fewest number correct.

Most research on heuristic methods has used the "thinking aloud" technique, as did Duncker (1945) in his seminal study. But some investigators have turned to other techniques. J. Davis (1964) devised a mathematical model based on solution time and used the model to demonstrate that problem solving occurs in stages related to the structure of the problem, much as Duncker had noted. Hayes (1965), using a "spy" problem in which subjects had to chain together previously-learned associations took acceleration of the solution process when subjects neared the goal as evidence of a planning heuristic. Anthony (1966) studied the relative effectiveness of working backward and working forward in a maze problem and found that subjects were able to shift to the direction that was more efficient for a particular problem.

One topic studied by Duncker that has received considerable attention is "functional fixedness" --the tendency for a previous use of an object in a given function to inhibit the object's being perceived in another function. Although Duncker's work with mathematics problems was more pertinent to education and although he himself admitted that in his monograph functional fixedness was discussed in detail out of proportion to its importance, the topic continues to inspire research, perhaps because of its bearing on the larger topic of rigidity in problem solving (see Cunningham, 1966).

The work of Raaheim (1965) illustrates the direction that some of the work on functional fixedness has taken. One can classify problems into two broad categories: those in which the goal is difficult to understand but easy to attain once understood, and those in which the goal is easily understood but difficult to attain. Raaheim found that performance on the first kind of problem was related to the ability to find many functions for a given object, to the ability to list many objects to fulfill a given function, and to general intellectual ability. Performance on the second kind of problem, however, was related only to the ability to list many objects to fulfill a given function. Raaheim's work implies the existence of several kinds of flexibility, whose role depends upon the structure of the problem and how it is seen by the subject. The larger question of how subjects adapt various heuristic methods to different kinds of problems remains virtually unexplored.

Additional References: Holden, 1964; Pylyshyn, 1963.

## Creative Behavior

Few studies of creative behavior have dealt directly with mathematics. Of these, most have been concerned with the construction of instruments for measuring mathematical creativity. Other studies, generally of more tangential relevance, have explored the relationship between creativity and problem solving.

### Measurement of Creativity

The view that intelligence tests are inadequate for the identification of creative talent has sparked attempts to construct new tests. Prouse (1967) reports on the construction of a ten-item test to measure divergent and convergent thinking in mathematics. Subjects were 312 seventh graders from 14 classes in five schools. Within-teacher correlations were calculated between the creativity test and measures of intelligence, achievement, school marks, preference for school subjects, and creativity as rated by the teacher. Correlations between the creativity test and the teacher ratings were low but both were moderately correlated with intelligence, achievement, and grade point average. The results were disappointing; they gave no indication that the creativity test was superior to other measures, including teacher ratings, in identifying gifted students.

A battery of creativity tests for grades five through eight was devised by Evans (1964), who started with 27 tests and reduced them to 16 on the basis of preliminary tryout and intercorrelation analysis. Final testing of 123 subjects of above-average mental ability showed few differences between mean performance at each grade, except for a general tendency for the fifth graders to score lower than the others. A composite score on the battery was significantly correlated with measures of intelligence, achievement, attitude toward mathematics, and general creativity (the latter consisting of an unusual uses test, an anagrams task, etc.). Evans' tests are clever, and though firm data on reliability and validity are lacking, the battery shows some promise as a measure of the ability to formulate new mathematical ideas.

Creativity and Problem Solving

Klein and Kellner (1967) studied differences in the performance of high and low creative subjects on a two-choice probability learning task similar to that used by Weir (1964). Subjects were 16 high and 16 low scorers, from a group of 130 male undergraduates, on the Remote Associates Test. An analysis of covariance, with IQ as the covariate, showed that high creative subjects took significantly longer in making a shift from one choice to the other and tended to match their responses with the objective probability of reinforcement sooner than did the low creative subjects. The results were interpreted as suggesting that the high creative subjects were demonstrating a greater tendency to form hypotheses about the pattern of reinforcements.

Additional evidence that high and low creative subjects approach problems differently came from studies by Eisenstadt (1966) and by Mendelsohn and Griswold (1964). Eisenstadt found that creative subjects were faster than noncreative subjects in solving rebus puzzles with incomplete information, in solving the puzzles after complete information was given, and in giving up on puzzles they could not not solve. Mendelsohn and Griswold found a positive relationship between creativity and the use of incidental cues in solving anagram problems.

Cicirelli (1965) reported relatively low correlations (0.11 to 0.26) for a group of 609 sixth graders between subscores on the Minnesota Tests of Creative Thinking (MTCT) and scores on the California Arithmetic Test. Of the three areas of achievement measured (arithmetic, reading, and language), arithmetic had the weakest relationship to creativity. Wodtke (1964) found that, whereas intelligence and problem-solving flexibility (as measured by the Luchins Water Jar Test) were correlated positively and significantly, flexibility was not correlated with creativity (as measured by the *MTCT*), even when the measures were corrected for attenuation. It appears that creativity, though it may be related to certain facets of problem solving, bears no simple relationship to problem-solving performance.

## Instructional Programs

Recent years have witnessed the development of increasingly sophisticated theory-based programs of instruction in problem solving and creative behavior. Though the majority of studies continue to be evaluations of a single device or technique, some attempts have been made to develop broader programs having an explicit theoretical rationale.

### Training in Heuristic Methods

Covington and Crutchfield (1965) report several studies with the *General Problem Solving Program* (*GPSP*), a well-conceived and apparently successful program they devised for teaching children to apply heuristic strategies to problems. Though the problems are not mathematical, the strategies are appropriate to mathematical problem solving. The program consists of a series of self-instructional booklets that present the continuing story of a brother and sister team, Jim and Lila, as they try to solve a series of puzzles and mysteries with the aid of their Uncle John, a high school science teacher and part-time detective. The pupil is supposed to identify with Jim and Lila as they gradually overcome their anxieties and become proficient in problem solving. The programmed booklets not only give the pupil repeated experiences in solving interesting problems, but they also show him strategies such as planning one's attack, searching for uncommon ideas, transforming the problem, and using analogies. Covington and Crutchfield found dramatic gains for an instructional group of fifth and sixth graders as compared with a control group on tests of problem-solving ability, tests of creative thinking, and attitude inventories. Five months after instruction, gains in problem-solving ability diminished somewhat but were still statistically significant; gains in creative thinking had become marginal (Covington, 1968).

An extensive tryout of the *GPSP* in 44 fifth-grade classrooms under conditions of minimal teacher involvement and a more rapid rate of presentation yielded similar, but less striking results (Olton et al., 1967). The performance of the instructed group was superior to that of the control group on 30 of 40 criterion measures of convergent and divergent thinking, but only 11 of these differences were statistically significant. Ripple and Dacey (1967) found, with a modification of the GPSP for eighth graders, no significant differences between instructed and control groups on four measures of verbal creativity. The instructed pupils solved the Maier Two-String Problem faster than did the controls, but the percentage of correct solutions was about the same.

Concerned over these apparent failures to replicate Covington and Crutchfield's findings and curious about the question of nonspecific transfer, Treffinger and Ripple (1968) investigated the effectiveness of the GPSP on verbal creativity, general problem solving, arithmetic problem solving, and attitudes, at each of the grades from four through seven. Various analyses were performed, including chi-square analyses of performance on individual problems and analyses of covariance with IQ and pretest scores as covariates. Only a few differences on the verbal creativity and problem-solving tests--not much more than would be expected by chance--reached statistical significance, and these differences formed no obvious pattern. (Although it should be noted that the results for arithmetic problem solving may have been clouded by the difficulty and the low reliability of the tests used.) Statistically significant differences favoring the instructed group were found on a measure of general attitudes about creative thinking and problem solving at all four grade levels. The results, as in the study by Olton et al., suggest that the GPSP may be most effective when presented at a slow pace with supplementary discussions by the classroom teacher. Also, although the GPSP may be successful in promoting some transfer to novel problems, unless the format of the problems resembles that of the training materials, transfer is likely to be minimal.

Daniels (1964) attempted to teach college students the heuristic method of "property analysis" (describing to oneself the elements in a task situation) in order to produce nonspecific transfer. He too was less successful than Covington and Crutchfield: significant increases in originality on Guilford's Unusual Uses Test were observed in one experiment but not in another, although the training conditions were identical, and no improvement was found in ability to solve a battery of insight problems. As Daniels pointed out, training programs emphasizing fluency of response may promote transfer to other divergent thinking tasks merely because the subject comes to expect that fluency is what is wanted. Such programs may well fail, however, in promoting performance on insight problems and other tasks in which fluency alone is of little help.

Attempts to promote transfer of heuristics from miscellaneous training tasks to problems from disciplines such as mathematics raise questions as to how general or how specific the heuristics should be. James Wilson (1967) predicted that subjects taught specific heuristics would perform better on training tasks but worse on transfer tasks than would subjects taught general heuristics. Subjects were trained on two theorem-proving tasks, one in symbolic logic and the other in elementary algebra, by means of self-instructional booklets. For each task, subjects were taught to use one of three kinds of heuristics: task-specific (applicable to the training task only), means-end (locating the key difference between the given situation and the goal and then searching for a means to reduce the difference), and planning (omitting details in the given situation and working out a proposed solution in general terms). A 3 x 3 x 2 factorial design was used, with three levels of heuristic for each task and two orders of task presentation. Dependent variables were derived from performance on the training tasks and on five transfer tasks (two similar and three dissimilar in format to the training tasks). Task-specific heuristics did not facilitate performance on the training tasks; in fact, on one training task the planning heuristic was superior to the others. On one of the (dissimilar) transfer tasks, the planning heuristic was superior to the others; otherwise there were no significant main effects. Significant interactions suggested that a combination of heuristics during training facilitated performance on some of the transfer tasks and that general heuristics learned in the first training task were practiced on the second task, thereby facilitating transfer.

Several researchers have recently sought, with little success, to improve problem-solving performance by giving the subject a model to follow. Brian (1966) taught college students to solve problems by using a flow chart that incorporated heuristics identified by Polya. No improvement was found in ability to construct mathematical models, to conjecture, and to use axioms, theorems, and algorithms, but some improvement was noted in ability to settle conjectures. The small sample size (17) and the lack of a control group prevented an adequate test of Brian's hypotheses; his analysis of processes deserves further investigation.

Post (1967) demonstrated that awareness of the types of mental operations thought to underlie problem solving does not necessarily improve performance. Ten seventh-grade mathematics classes constituted the sample. The classes were matched on problem-solving ability, and while the control group followed the regular schedule, the experimental group was given a three-day introduction to a list of processes drawn from the literature on problem solving followed by six weeks of practice in using the processes. Analysis of gain scores on a problem-solving test showed no significant difference between the experimental and control groups.

Additional Reference: Means and Loree (1968).

## Learning by Discovery

Mathematics seems to have a natural affinity with learning by discovery: mathematics curriculum reformers such as Max Beberman and Robert Davis have been among the most articulate spokesmen for the view that children should discover some of what they learn; and on the other hand, psychologists studying discovery learning have often used mathematical tasks. As Wittrock (1966) notes, in a comprehensive analysis of the literature, research on discovery learning generally suffers from conceptual and methodological weaknesses and has yielded inconsistent results. It is encouraging, therefore, that several recent studies, better designed than most, have demonstrated some superiority of discovery methods for promoting transfer of problem-solving skills.

One way of contrasting "discovery" and "expository" methods is in terms of the sequential organization of instruction. In a discovery method, verbalization of a concept or generalization comes at the end of the instructional sequence; in an expository method, it comes at the beginning. Using this contrast, Worthen (1968) studied the effects of six weeks of instruction in mathematical concepts by the two methods. In each of seven schools, one sixth-grade class used a discovery method and one used an expository method; in an eighth school, one fifth-grade class used each method. The same teacher taught both classes in each school. Worthen attempted to control a number of variables left uncontrolled in previous studies. For example, all teachers participated in a training program on the use of the methods and materials, and then, as a check that the methods were being followed, teacher behavior was rated by observers during the experimental period and by the pupils at the end of the period. Ratings showed close fidelity to models of each method that Worthen had delineated. Analysis of covariance, using IQ scores, Metropolitan Achievement Test arithmetic computation and problem-solving subscores, and pretest scores on a concept knowledge test as covariates, indicated that the expository group showed more initial learning than the discovery group, contrary to Worthen's expectation. When initial learning was taken as a covariate, however, the discovery group showed greater retention of concepts after five and eleven weeks and an indication ($p < .08$) of greater transfer of mathematical principles to novel situations. The discovery group also made greater gains on oral and written tests of discovering a short cut for working a series of problems.

Wills (1967) investigated the effect of learning by discovery on problem-solving ability. Two groups of eight intermediate algebra classes were given two weeks' instruction on figurate numbers and recursive definitions. The instructional materials, in a workbook format, introduced a topic by presenting a difficult problem that required a generalization, guiding the student with a series of simpler problems, prompting the student to look for a pattern in the problems, and giving the student a check on his generalization. In one group of classes, the teachers discussed various heuristic methods for discovering the generalization in the instructional materials; in the other group, the teachers gave no such guidance. Before and after instruction, both groups took a 60-item test on mathematical topics not covered in the unit. On the posttest both groups doubled their pretest performance, whereas a control group that took the tests without the intervening instruction made only a minor gain. Guidance by the teacher on heuristic methods apparently did not contribute to the gains; the adjusted means of the two experimental groups on the posttest did not differ significantly.

Scandura (1964) demonstrated, in three small studies, that variations within discovery and expository methods, such as the directness of presentation and the point at which a generalization is introduced, may be responsible for some of the conflicting results in earlier research. In one experiment, a discovery method yielded an advantage in nonspecific transfer, but in the two other experiments, under modified conditions, it did not. Scandura concluded that the timing of instructional steps in discovery learning is one of the most critical features. Another small-scale study, by Meconi (1967), found no significant differences on either a transfer or a retention test among "pure discovery," "guided discovery," and "rule and example" methods for teaching summation of sequences. Both Meconi and Scandura used instructional treatments of only a day or so's duration and with small numbers of students, so their results are not directly comparable with those of Worthen and Wills.

### Training in Verbal Skills

Since students frequently have trouble reading verbal problems in mathematics textbooks, it is natural to ask whether training in verbal skills might improve problem-solving performance. Reasoning that knowledge of vocabulary is an important component of problem-solving ability, VanderLinde (1964) tested whether the study of vocabulary lists would result in gain on arithmetic problem-solving tests. Nine fifth-grade classes, matched with nine control classes on IQ and on Iowa Tests of Basic Skills (ITBS) subtests of vocabulary, reading comprehension, arithmetic concepts, and arithmetic problem solving, studied a different list of eight quantitative terms every week over a period of 20 to 24 weeks. On a readministration of the ITBS, the experimental group exceeded the control group on both arithmetic subtests. Further analysis of the arithmetic problem-solving subtest revealed no differences between the sexes but a significantly lower mean gain by the low IQ group compared with groups of average and above-average IQ.

Irish (1964) reported a two-year investigation in which a group of nine fourth-grade teachers in small city school system were asked to spend part of the time in class ordinarily spent on computation helping pupils state generalizations about number operations. In both years, pupils of these teachers made greater gains on problem solving as measured by the Sequential Tests of Educational Progress Mathematics Test and on computation as measured by the School and College Ability Tests than pupils of other fourth-grade teachers in the system.

Additional References: Call and Wiggin (1966); Lyda and Duncan (1967).

### Special Methods and Devices

Much attention has been focused in recent years on finding methods and devices that would improve problem solving without putting the child in the kind of straitjacket provided by formal analysis and other prescriptive techniques. The traditional approach to the solving of word problems in the elementary grades has been characterized as a "wanted-given" procedure--the child is taught to ask himself, "What is wanted?" and "What is given?" and then to perform the appropriate operations on the data to yield values for the unknowns. In several recent studies, other approaches have been pitted against this traditional one.

John Wilson (1967) contrasted one version of the wanted-given approach, in which the child analyzes structural relationships between the data and the unknowns, with an "action-sequence" approach, in which the child looks for the operations suggested by the sequence of actions in the problem. In both approaches, the child writes and solves a number sentence that expresses the structure of the problem. Fifty-four fourth-grade subjects at three levels of mental ability were randomly assigned to a wanted-given, an action-sequence, and a control group. Each group was given three periods of instruction a week for nine weeks, using worksheets supplemented by instruction from the teacher (except for the control group, which just worked the problems and then spent the remaining time on other activities). A 3 x 3 factorial analysis of variance followed by multiple comparison tests showed that on measures of ability to choose correct operations, of ability to solve problems, and of speed in solving problems, the wanted-given group performed better than the other two groups after three, six, and nine weeks of instruction, and again nine weeks after instruction had ended. Wilson's study provides strong support for the superiority of the wanted-given approach, although as Zweng (1968) observes, Wilson's wanted-given treatment varies considerably from the traditional one.

Lerch and Hamilton (1966) compared a structured-equation approach, in which pupils wrote a number sentence expressing the relationships in the problem (apparently something like Wilson's wanted-given treatment), with a traditional wanted-given approach, in which no number sentence is written. After five months of instruction, a fifth-grade class taught according to the structured-equation approach showed greater gain than a class taught according to the wanted-given approach on a measure of ability to choose correct operations but not on a measure of ability to solve problems.

The effects of varying amounts of attention during arithmetic instruction to the structure of the whole numbers versus verbal problem solving was studied by Stuart (1965). Fourth-grade pupils in ten schools were assigned at random to one of three program treatments: (1) a program designed to teach basic concepts of numbers and computation, with no solving of verbal problems; (2) a program designed to teach the analysis of verbal problems, how to express the relationships as number sentences, and how to solve the sentences; and (3) a program consisting of equal parts of the other two programs.

Analysis of covariance, with IQ and subscores on the *Stanford Arithmetic Test* (*SAT*) as covariates, showed significant differences in posttest performance on the problem-solving section of the *SAT* but not on the computation or concept sections. A delayed administration of a third form of the *SAT* yielded no significant differences. Stuart concluded that for immediate problem-solving performance, the "mixed" program was superior to the program that emphasized structure of whole numbers and at least as good as the program that emphasized problem solving alone.

Riedesel (1964) devised a series of problem-solving lessons, at two levels of difficulty, in which pupils were instructed and given practice in writing number sentences, drawing figures, formulating problems, presenting problems orally, and solving nonnumerical problems. Eleven sixth-grade classes were given the experimental instruction for ten weeks. When compared with nine control classes that got no special instruction in problem solving, the experimental classes made significantly greater gains on a problem-solving test constructed for the study and gains approaching significance on the Arithmetic Problem-Solving subtest of the *Iowa Tests of Basic Skills*.

The hypothesis that experience in writing and solving one's own problems is more beneficial than practice in solving textbook problems was tested by Keil (1964). Pupils in four sixth-grade classrooms spent one period a week for 16 weeks writing and solving problems about a given situation, while pupils in a control group of four classrooms solved textbook problems about the same situation. Results of an analysis of covariance, with an IQ and pretest scores on the *Sequential Tests of Educational Progress* (STEP) Mathematics Test as covariates, and scores on alternate forms of the *Metropolitan Achievement Test* and *STEP* as dependent variables, supported Keil's hypothesis.

Other dimensions of the kind and amount of problem-solving practice that children should receive were explored by Koch (1965), who found in a small, but carefully done study that increasing the amount of homework assigned did not improve problem-solving performance, and by Traub (1966), who found that performance on a complex task (addition on the number line) was facilitated more by heterogeneous than by homogeneous subtasks in a self-instructional program.

Two studies dealt with techniques for teaching problem solving at the ninth-grade level. Denmark (1964) compared a "deductive" approach (arranging data and unknowns in a table and then reading off the appropriate equations) with an "inductive" approach (looking for a pattern in successive trials). On a posttest, students taught the inductive method were better able to arrive at correct solutions, but when credit was given for formulating correct equations and solving them correctly, the students taught the deductive method excelled. Bechtold (1965) found that including problems having extraneous data along with other assigned problems in algebra classes yielded superior performance, not only on extraneous data problems, but on other problems as well.

In one of the few studies with college students, Gangler (1967) assessed the effects of overt responding, motivation, time of day, intelligence, and mathematical background on the learning of symbolic logic tasks. In addition to his findings on motivation mentioned in an earlier section, Gangler found that subjects who wrote answers to exercises rather than responding covertly made fewer errors on problem-solving tasks. Overt responding during learning was particularly beneficial for students whose motivation was raised by telling them that their performance would count toward their grade. Students who had taken relatively few mathematics courses in high school and college did better on learning tasks when their responses were covert rather than overt. Gangler found a number of significant higher-order interactions too complex to interpret and summarize here but suggesting the existence of complicated relationships among personality and task variables in programs to develop problem-solving ability.

Additional References: Allen, Allen, & Miller (1966); Clark (1967).

## Effects of Curriculum Programs and Class Organization

An assumption explicit or implicit in most modern mathematics programs is that the innovations will result in, among other things, improved ability to solve problems. Two studies provide partial tests of this assumption. Scott (1965) reported data that he interpreted as evidence that modern programs prepared children somewhat better for coping with "insoluble" word problems than do traditional programs. Second and third graders in classes using School Mathematics Study Group (SMSG) texts and texts based on the Greater Cleveland Mathematics Project materials, and fifth and sixth graders in classes using SMSG texts, were compared with children in the same grades who had used traditional texts. Analyses of covariance, with IQ as a covariate, yielded significant differences, favoring the modern text groups, only at grades two and five. The lack of consistency in these results, together with the low performance overall on the insoluble items, suggests that modern programs do not materially improve the skill of dealing with inconsistencies.

The hypothesis that instruction in concepts of set theory improves problem-solving performance was tested by Smith (1968). Two seventh-grade classes received twenty days of instruction in set theory while two other classes studied the fundamental arithmetic operations. The students instructed in set theory gained more in logical reasoning ability but not in ability to solve percentage problems than did the other students.

Individually Prescribed Instruction (IPI) has received considerable attention as a technique for organizing instruction. Deep (1966) found no significant differences, after adjustment for initial performance, in computation or problem-solving scores among students of high, average, and low ability at grades four, five, and six after a year of IPI instruction. He concluded that standardized tests are inappropriate for measuring achievement in an IPI program, which may be true, but other evidence (Lipson, 1967) indicates a general lack of correlation between IQ and progress under IPI.

Hudgins and Smith (1966) reported that when a small group of elementary school children engaged in cooperative problem solving, the group performed better than its most able student in solving arithmetic problems if and only if he was not perceived by the group as being most able. Problem solving by groups seems not to be a widespread practice in mathematics classrooms, but Hudgins and Smith's findings on the importance of sociometric status factors may have larger implications for the pattern of interactions and pupil contributions during a problem-solving session.

*Additional Reference*: Moore and Cain (1968).

## Teacher Influences

Polya (1962, 1965) has argued that teachers cannot teach problem solving unless they have had some experience themselves in solving problems. Though this argument has not been given a direct test, Godgart (1964) has demonstrated that, at least by one measure, teachers' problem-solving ability in mathematics is not related to pupil progress. The Mathematics Test of the *Sequential Tests of Educational Progress* (*STEP*) was administered to 35 fourth-grade teachers, and the Arithmetic Problem-Solving subtest of the *Iowa Tests of Basic Skills* (*ITBS*) was administered twice to their pupils--at the beginning of the fourth grade and again at the beginning of the fifth grade. The teachers performed significantly better on the *STEP* than did the norming population of college sophomores. When the teachers were divided into five equal groups according to their performance on the STEP, and analysis of covariance performed on class means with the ITBS pretest as the covariate and the ITBS posttest as the dependent variable, the groups did not differ significantly. Furthermore, teachers' problem-solving ability as measured by the STEP was unrelated to such background measures as age, tenure status, undergraduate major, and number of mathematics content and methods courses taken.

Broome (1967) reported an unsuccessful attempt to relate teachers' creativity to children's learning. Six low-creative and six high-creative teachers, as measured by the Minnesota Tests of Creative Thinking (MTCT), were selected at random from the fifth-grade teachers in a city school system. Seventy-one pupils of the high-creative teachers and 71 pupils of the low-creative teachers were administered MTCT Form A in the fall and MTCT Form B in the spring. Achievement data were obtained from school records. Estimated true-gain scores showed no differential change in creativity, vocabulary development, reading comprehension, or arithmetic reasoning for pupils of the two groups of teachers.

If the teacher does have an impact on pupils' problem-solving ability and creativity, the locus of this impact must be the classroom. Stilwell (1967) adapted the Flanders interaction-analysis scheme to study problem-solving activity in geometry classrooms. One of Stilwell's most interesting findings concerned the relatively small amount of class time (less than 3 percent of all problem-solving activity) spent in discussing a method for solving a problem. Looking back at a problem or ahead to its implications occupied 7 percent of the problem-solving activity, with teacher requests to look back at the solution correlated positively and significantly with greater teaching experience.

## Conclusion

A good share of the research in mathematics education, now as in the past, is being done by doctoral students. Though many theses on problem solving and creative behavior in mathematics are of a quality and sophistication that surpasses the general level of journal articles on these topics, the theses are relatively inaccessible. The forthcoming Journal for Research in Mathematics Education should go far toward alleviating this situation.

As research in mathematics education becomes more sophisticated in design, more explicitly grounded in theory, and more closely allied to developments in other fields, the mathematics educator's one-shot comparisons of ill-defined "methods" and the psychologist's laboratory studies of arbitrary, highly artificial concepts should give way to diagnostic, long-term studies of learning and thinking in school settings. Evidence of this trend can be noted in the studies reviewed, as can a general movement toward more complex designs and analyses.

Unfortunately, the increasing complexity of design has been accompanied by an increasing number of methodological blunders, such as the inappropriate use of analysis of covariance and the use of subjects as experimental units when intact classes have been assigned to treatments. More disturbing still is the investigators' apparent ignorance that statistical assumptions are being violated. Mathematics educators, of all people, should be highly skeptical about the congruence between an analytical model and the "real-world" data.

Much has been said lately about the need for large-scale, complex studies in mathematics education, but the researcher--most likely a doctoral student--who chooses to investigate problem solving and creative behavior in mathematics is probably best advised to undertake clinical studies of individual subjects (children gifted in mathematics, children for whom mathematics is particularly difficult, etc.), not only because clinical studies are more commensurate with limited financial and administrative resources, but also because our ignorance in these areas demands clinical studies as precursors to larger efforts.

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